PRACTICAL ASTRONOMY

AND

GEODESY.

Louis B. Stewart.
Toronto 1889

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A COURSE

—OF—

PRACTICAL ASTRONOMY

FOR SURVEYORS

—WITH—

THE ELEMENTS OF GEODESY

—BY—

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PREFACE.

This manual has been drawn up for the use of the Cadets of the Royal Military College of Canada. The first five chapters on Practical Astronomy embrace that portion of the subject with which all Land Surveyors in this country ought to be familiar. The remaining chapters, together with the part of the work which treats of Geodesy, touch on the more important parts of the additional course, as regards those subjects, laid down by Government for candidates for the degree of Dominion Topographical Surveyor. It has become absolutely necessary to draw up some compilation of this kind, because, while many of the Cadets are anxious to qualify themselves as far as possible in the above-mentioned course, the number of different books they would have had to refer to in order to obtain the requisite knowledge would have entailed on them a heavy expense. In order to make the work as cheap as possible the number of diagrams has been cut down to a minimum, it being intended to supply the place of expensive plates of instruments *et cetera* by lecture illustrations. The author has also made the higher portion of the Astronomical course

as brief as possible. It will be found treated in the fullest manner in Chauvenet's Astronomy.

Geodesy being both a difficult and a very extensive subject no attempt has been made to write anything like a treatise on it. All that has been aimed at has been to give a sketchy account of its most salient points, adding a few details here and there. The student who wishes to pursue the subject further is referred to standard works, such as Clarke's Geodesy.

The author has to acknowledge having made more or less use of the following:

Chauvenet's Astronomy, Puissant's Géodésie, Clarke's Geodesy, Frome's Trigonometrical Surveying, Loomis' Practical Astronomy, Gillespie's Higher Surveying, Deville's Examples of Astronomic and Geodetic Calculations, the U. S. Naval Text Book on Surveying, and Jeffers' Nautical Surveying. He has also to thank Lieut.-Colonel Kensington, R.A., for valuable assistance in investigating some doubtful formulas.

KINGSTON, CADADA,
January, 1883.

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NOTE TO PAGE 52.

By drawing a figure it can be easily shown that, in the case of a horizontal dial, if φ is the latitude, P the hour angle, and α the angle the corresponding hour line makes with the meridian line, then :

$$\sin \varphi = \cot P \tan \alpha$$
$$\text{or } \tan \alpha = \sin \varphi \tan P.$$

Similarly, in the case of a dial on a vertical wall facing south,

$$\tan \alpha = \cos \varphi \tan P.$$

In the latter case the angle α is measured from a vertical line on the wall. The stile is, of course, set parallel to the polar axis.

We can thus find the hour lines for each hour, for any given latitude, by solving these equations.

Louis B. Stewart.

PART I.

PRACTICAL ASTRONOMY.

CHAPTER I.

GENERAL VIEW OF THE VISIBLE UNIVERSE. THE FIXED STARS. THE SOLAR SYSTEM. APPARENT AND REAL MOTIONS OF THE HEAVENLY BODIES. DIFFERENT METHODS OF RECKONING TIME.

The visible universe, outside our earth, comprises the sun, moon, planets, fixed stars, milky way, nebulæ, shooting stars, and the zodiacal light, besides an occasional comet.

The comets, shooting stars, and zodiacal light will not be further alluded to here. The milky way and the nebulæ (white cloudy patches) when examined with powerful telescopes generally resolve themselves into clusters of separate stars; a few nebulæ, however, still retaining their cloud-like appearance.

The fixed stars, as they are called, are doubtless suns, scattered irregularly (or more properly in clusters) through

space. They are classified by astronomers into magnitudes, the brightest being those of the 1st magnitude. Those of the 6th magnitude are about the smallest visible to the naked eye, those below that size being only visible through telescopes. Although differing so much in brightness, the most powerful telescope fails to show them of any measurable size, and they all appear mere points of light. Their brightness, as seen by us, depends, probably, partly on their distances, partly on their size, and partly on their natural brilliancy, while that of a few of them varies at regular intervals. The colour of the stars also varies, inclining in some to white, in others, to red, blue, or green. Some stars are connected in pairs and revolve round a common centre. It has been ascertained by the spectroscope that the elements present in the sun and stars are identical with those composing our earth ; at least no new ones have yet been discovered.

The stars were grouped by the ancients into constellations, of which the Great Bear and Orion are instances; and a number of the most remarkable stars received special Arabic names, *e.g.*, Arcturus and Aldebaran. The stars composing a constellation are catalogued according to their brightness, the Greek letters being used to distinguish them. Thus Aldebaran is α Tauri, and the two stars of that constellation next in brightness are β and γ Tauri. When the Greek alphabet is exhausted English letters are used, and finally numbers. Thus we have *h* Virginis and 51 Cephei. The stars are numbered, not according to their brightness, but in the order of their right ascension.

The distances of the fixed stars from the earth and from each other are so great as to be almost beyond human conception. It was for long believed that they could not be measured. It was, however, eventually found that in the case of some of them, by taking a line through space

joining opposite points of the earth's orbit as a base, and the star as the apex of a very acute-angled triangle, the angles adjacent to the base could be measured and the acute angle thus determined. The length of the base being known gives the star's distance. To give an idea how far off the nearest star is it may be mentioned that a ray of light would pass round the earth (about 24,900 miles) in a quarter of a second; it takes $8\frac{1}{4}$ minutes to traverse the 93 millions of miles from the sun to the earth; and $3\frac{1}{4}$ years to reach us from the star. And yet, could we be transported to that star, we should still see all the other familiar constellations and stars apparently in exactly the same positions as we see them here. So vast are the distances that the change of position of the observer would have about as much effect on that of the stars as would an interchange of two adjoining grains of sand on a large table covered with them.

The nearest star, as at present known, is α Centauri, which is 200,000 times farther off than the sun. The approximate distances of a few others, in terms of the number of years it takes their light to reach us, are as follows:

β Centauri	$6\frac{1}{2}$	years.
61 Cygni	8	"
Sirius	16	"
Procyon	16	"
Arcturus	16	"
Vega	16	"
Pole Star	32	"

About 100,000 stars have been catalogued altogether. The number visible with the naked eye is about 15,000. In latitude 50° north only about 2,000 can be thus seen at any one time.

Our sun is only one of the stars, and the latter, though called "fixed," are in reality all moving according to the laws of dynamics. What these motions are we cannot tell, as we do not yet know the manner in which the

masses are distributed through space. It has however been ascertained, not only that they are slowly changing their position with regard to each other, but that in one part of the heavens they are getting farther apart, thus indicating that the motion of our sun with his attendant planets is in the direction of that part. It may be inferred that the stars are, as a rule, the centres of planetary systems like our own, and that possibly each system has at least one planet in a state of development permitting of its habitation by living creatures. Our own solar system consists of the sun and eight planets, besides a number of small planets revolving between the orbits of Mars and Jupiter. The planets move in ellipses, of which the sun is a focus. Several of them have moons or satellites, and all, including the sun, revolve on their own axes.

The planes in which the larger planets move all nearly coincide with that of the earth, the greatest difference being in the case of Mercury, whose orbit is inclined to ours at an angle of 7°. Looking down on the plane of the system from its northern side, the direction of the motion of the planets round the sun, of the satellites round the planets, and of all its members (including the sun) round their own axes, would be seen to be opposite to that of the hands of a clock. The earth has one moon (distant from it about 60 radii of the earth), Mars two, Jupiter four, Saturn eight, Uranus four, and Neptune one. The distances of the planets from the sun are nearly in the following proportion: Mercury 1, Venus 2, the Earth 2.6, Mars 4, Jupiter 13, Saturn 25, Uranus 50, Neptune 80. The density or specific gravity of Mercury is a little more, and that of Venus a little less, than that of the earth, of the moon $\frac{3}{5}$ths, of the sun and Jupiter one quarter, of Mars $\frac{7}{10}$th, Saturn $\frac{1}{9}$th, Uranus $\frac{1}{5}$th, and Neptune $\frac{1}{7}$th.

To compare their relative sizes; if we took a globe four feet in diameter to represent the sun, the moon would be about the size of a grain of shot, Mercury of a buckshot,

Venus and the Earth a small spherical rifle bullet, Mars a small revolver bullet, Jupiter an 18-pounder round shot, Saturn a 9-pounder round shot, and Uranus and Neptune large grape shot; the latter the largest. The mass of Jupiter is 300 times that of the Earth, while that of Mercury is only about $\frac{1}{15}$th, Mars $\frac{1}{10}$th, and the Moon $\frac{1}{80}$th.

The planets are easily recognized by their changing their places in the sky relatively to the fixed stars—hence their name, which means "wanderer." They may also be known by their shining with a steady fixed light instead of twinkling. They shine as a rule only by the sunlight reflected from their surfaces, and when viewed through a good telescope, look like small moons, instead of mere points of light, as in the case of the fixed stars, and may be noticed also to pass through phases like the moon, especially in the case of the two that are inside the earth's orbit. The variability of their brightness is caused partly by this, partly by change in their distances from the earth. The well-known rings of Saturn are now supposed to consist of a shower of meteorites revolving round him.

Supposing us to be situated in the northern hemisphere, and not too far north, if we watch the apparent motions of the heavenly bodies in the sky we shall notice the following facts. The sun rises latest and sets earliest about the 21st of December, while the opposite is the case about the 21st of June. During the winter half of the year his rising and setting is south of the east and west points of the horizon, and during the summer half they are north of it; while at two intermediate periods, known as the equinoxes, he rises due east, remains in sight for 12 hours, and sets due west. At midwinter the arc he describes through the sky is the lowest, and at midsummer the highest.

When the moon is first seen as a young moon she is a little to the east of the sun. She rapidly moves through

the sky towards the east, so that about full moon she rises as the sun sets, and later on is seen as a crescent rising before the sun in the early morning. ~~The height to which she rises in the sky will be observed to be (unlike the case of the sun) quite independent of the time of year.~~ The interval between two new moons~~—that is the time she takes to make an apparent circuit of the sky~~—is about 2~~8~~9 days; and she rises each day about ~~three quarters of an hour~~ later than the day before. about [illegible] on an average

The stars, if carefully observed, will be noticed to rise each night a little less than four minutes earlier than they did the night before, so that at any given hour a certain portion of the sky which was visible at the same hour the night before will have disappeared in the west, and a similar portion will have come into view in the east. In fact the whole mass of the stars appears to be slowly overtaking the sun (or rather the sun to be moving through the stars); and, as a consequence, if the stars were visible in the day time this motion could be plainly seen. The points of rising and setting of the stars are always the same. The sun and all the stars reach their greatest height in the sky—or culminate, as it is termed—at a point where they are due north or south of the spectator.

The stars in the northern portion of the sky, from the horizon up to a certain point depending on the position of the observer, never rise or set, but describe in the twenty-four hours concentric circles round an imaginary point called the pole, and in a direction contrary to that of the hands of a watch.

The different planets, if carefully observed, will be noticed, not only to change their positions among the fixed stars, but to vary in brightness from time to time.

So much for the apparent motions of the heavenly bodies. We have now to consider their real ones.

The earth describes an elliptic orbit round the sun in about 365¼ days. It also revolves on its own axis in about a day. This axis remains parallel to itself and is inclined to the plane of the orbit at an angle of about 23° 27′. Hence the phenomena of the seasons, and of the varying positions of the sun from day to day.

Fig. 1.

Figure 1 shows the position of the earth with reference to the sun at the different seasons. N is the north pole, S the south pole, and A a point in the northern hemisphere. The left hand sphere shows the earth's position when it is midwinter at A, and the right hand sphere when it is midsummer.

The motion of the earth round the sun causes the latter to continually change its apparent position amongst the stars. Its path through them is called the ecliptic, and lies, of course, in the plane of the earth's orbit. The earth's revolution round its own axis, although on an average 24 hours if taken with reference to the sun, really takes place in space in about 3 minutes and 56 seconds less than 24 hours, the difference being, in fact, the same as that between two successive risings or settings of the same star. It should also be noted that, owing to the enormous distances of the fixed stars from us, all lines drawn from the earth, no matter what its position, to any star, are sensibly parallel.

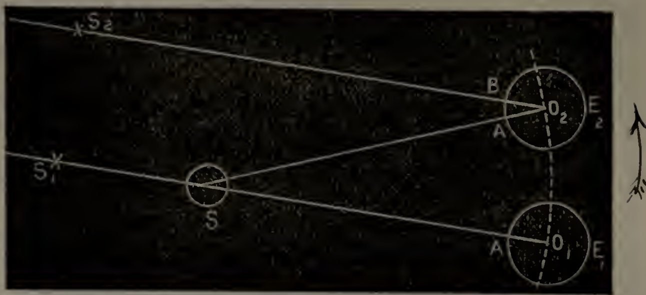


Fig. 2.

In figure 2 S is the sun and E_1, E_2 the position of the earth on two successive days when it is apparent noon at the point A. O_1 and O_2 are the earth's centre, and S_1 S_2 the apparent positions of a certain star as seen from A, O_1 S_1 being parallel to O_2 S_2. Let S_2 O_2 intersect the earth's circumference at B. It is evident that when the earth's revolution on its axis in the direction of the arrow has brought A to the position B in E_2 that it will have described a complete revolution with reference to space and that the star will again be on the meridian—in other words that a sidereal day will have elapsed since it left the position E_1; and that to bring the sun on to the meridian at A in E_2 it will have to describe an additional arc B A. This arc is the same as the angle B O_2 A, which is equal to the angle O_2 S O_1, and is the difference between a solar and a sidereal day. If there are n days in the year the value of this arc will be $\frac{360°}{n}$. Reduced to time it is about 3m. 56s.

Ordinary clocks are arranged to keep such a rate that 24 of their hours give the average interval between two successive culminations of the sun. The real interval is, however, sometimes more, sometimes less than this ; and, consequently, the sun does not culminate or pass the meridian at noon, but sometimes before it, sometimes after it, the greatest difference being about 16⅓ minutes. The instant the sun is on the meridian is called "apparent

noon." Noon as shown by a perfect clock is called "mean noon." The interval between the two is called the "equation of time." Its greatest amount is about the 1st of November, when the sun culminates about 11h. 43m. 41s. A.M. The equation then diminishes till about the 24th December, when mean and apparent noon coincide. After that the equation increases (the sun culminating after noon) till it attains a maximum of 14½ minutes about the 11th February, and then continues to decrease, becoming zero again about the 14th of April. It attains a maximum of 3m. 50s. about 14th May, becomes zero about 14th June, 6¼ minutes about 25th July, and zero 31st August. The cause of the equation of time is as follows. If the earth moved round the sun in a circle and at a uniform rate, and if the axis on which it itself turns were perpendicular to the plane of its orbit, the sun would culminate each day at noon exactly. But the earth moves in an ellipse and at a variable rate, and its axis is inclined to the plane of the ecliptic at a considerable angle, the combined effect being that we have the equation of time.

The great circle on the earth whose plane passes through the centre and is at right angles to the axis is called the "equator," and the projection of its plane in the heavens is also called the equator, and sometimes the equinoctial. If the sun, in its apparent annual path, moved at a uniform rate and traversed the equinoctial instead of the ecliptic we should have no equation of time. An imaginary sun moving in this way is called the "mean sun."

In addition to the time kept by an ordinary clock and that kept by the sun—in other words "mean time" and "apparent solar time"—we have a third kind called "sidereal time," that is, the time kept by the stars. It has been already mentioned that the interval between two successive culminations of the same star is a little less than 24 hours; the time it takes, in fact, for the earth

to make a single revolution on its axis. If we divide this interval into 24 equal parts we have 24 sidereal hours; and if we construct a clock with its hours numbered up to 24 instead of 12, and rate it to keep time with the stars, it is easy to see that the hour it shows at any instant will give the exact position of the stars in their apparent diurnal revolution round the earth. Clocks and chronometers of this description are used—the former in fixed observatories, the latter for surveying purposes.

The subject of sidereal time will be referred to later on. Before proceeding further it will be necessary to explain the meaning of the various astronomical terms in ordinary use.

CHAPTER II.

EXPLANATION OF CERTAIN ASTRONOMICAL TERMS. THE NAUTICAL ALMANAC.

For practical purposes the earth may be considered as a stationary globe situated at the centre of a vast transparent sphere at an infinite distance to which are attached the fixed stars, and which revolves round it in a little less than 24 hours. The sun, moon, and planets appear to move on the surface of this great sphere, the sun in the ecliptic, the rest in their respective orbits.

The extremities of the earth's axis are called the poles; and the poles of the great sphere are the points where the axis produced meets it.

Great circles passing through the poles are called "meridians." This term applies both to the earth and the great sphere. In the case of the latter they are also called "declination circles." Meridians are also called "hour circles," and the angle contained between the planes of any two meridians is called an "hour angle," because it is a measure of the time the sphere takes to revolve through that angle. It follows that the hour angle is the angle formed by two meridians at the poles.

In speaking of the meridian of a place we mean the great circle passing through the place and the poles; and a great circle passing through the poles of the great sphere and the zenith (or point in the sky immediately

over the observers head) is the meridian for the instant, as regards the great sphere.

To fix the relative position of points on the earth's surface we employ certain co-ordinates, called "latitude" and "longitude." The former is the angular distance of any point from the equator, and is measured along a meridian north or south as the case may be. The latitude thus varies from zero at the equator to 90° at the poles.

Longitude is the angular distance of the meridian of the place from a certain fixed initial meridian, and is measured either by the intercepted arc of the equator or by the angle contained by the two meridians. Longitude is measured east for 180° and west for 180°. Different countries reckon from different initial meridians. The English use that of Greenwich. The present system has many inconveniences, and it is to be hoped that some day the world will unite in adopting some fixed meridian and will reckon longitude through the whole 360 degrees instead of as at present.

The position of the heavenly bodies on the great sphere is determined by similar co-ordinates, but the latter are called "declination" and "right ascension," the former corresponding to latitude and the latter to longitude. Declination is measured from the equinoctial towards the poles, and right ascension eastward from a certain meridian. The latter is, however, reckoned through the whole 360°, and is counted by hours, minutes, and seconds instead of by degrees, 1 hour corresponding to 15 degrees.

The point where the zero or 24-hour meridian cuts the equator is called the "first point of Aries," and is designated by the symbol ♈. It is also one of the intersections of the equator with the ecliptic. On referring to the Nautical Almanac it will be seen that the co-ordinates of the stars are continually changing. The fact is that, owing to the slow conical motion of the earth's axis known as the "pre-

cession of the equinoxes," the planes of reference are changing. This, however, causes no practical inconvenience, as the relative positions of the stars remain the same.

It should be noticed here that the terms "latitude" and "longitude" are also used with reference to the heavenly bodies, and are liable to cause confusion. These co-ordinates are measured from and along the ecliptic, and are not required for the problems here treated of.

Besides the above-mentioned co-ordinates which relate to the relative position of points on a sphere another set is necessary to fix the position of a heavenly body with reference to the observer at any instant. They are called "altitude" and "azimuth." The first scarcely needs explanation. The second is the angle formed by the vertical plane passing through the observer and the object with the plane of the observer's meridian. The altitude and azimuth of a star at any instant are, in fact, the angles read by the vertical and horizontal arcs of a theodolite respectively when the latter has been clamped with its zero due north, and the telescope has been directed on the star. Azimuth is generally reckoned from the north round by the east, south, and west; but it is sometimes reckoned from the south.

The plane of the "sensible horizon" is the horizontal plane passing through the observer's position, and therefore tangential to the earth's surface at that point. The "rational horizon" is a plane parallel to that of the sensible horizon and passing through the centre of the earth. The projections of these two planes on the great sphere coincide, being at an infinite distance.

It is easy to see that about half the great sphere is in sight at any instant. The portion that is visible depends generally on the latitude of the place and the sidereal time of the instant. At the north pole the whole northern hemisphere would be always in sight and no other

part. At the south pole the view would be limited to the southern hemisphere. At the equator both poles would be on the horizon, and every point on the great sphere would come in sight in succession. At intermediate places a certain portion round one pole would always be above the horizon, while another portion round the other pole would never be visible.

"Parallels of latitude" are small circles made by the intersection with the earth's surface of planes parallel to the equator. Similar circles on the great sphere are called "declination parallels." A little consideration will show that within a certain distance of the equator at each side of it the sun will, twice in the year, pass overhead at mid-day. The belt enclosed between the two parallels within which this takes place is known as the "tropics."

A few more technical terms require explanation. When speaking of the "hour angle" of a heavenly body at any instant we mean the angle formed at the pole by the meridian circle of the instant and the declination circle passing through the body.

By the term "circumpolar star" is meant a star which never sets but appears to describe a complete circle round the pole. These stars cross the merdian twice in the twenty-four hours. One crossing is called the "upper transit," the other the "lower transit." At the points between the transits at which the stars have the greatest azimuth from the meridian they are said to be at their "greatest elongation," either east or west.

The words "transit" and "culminate" have the same meaning when used with reference to stars which rise and set.

"Parallax" is the change in the apparent relative position of objects owing to a change in the observer's position. Astronomically it generally signifies the difference in the apparent position of a heavenly body as seen by an

observer from what it would be if viewed from the centre of the earth. Parallax is greatest when the object is on the horizon, and nothing when it is in the zenith. The moon, from being near the earth, has a considerable parallax. That of the sun does not exceed 9″. The positions of the sun, moon, and planets given in the Nautical Almanac are those which they would have as seen from the earth's centre, and it is therefore necessary to correct all observations on those bodies for parallax.

Parallax causes the object to have less than its true altitude. Refraction has the opposite effect. The latter, like the former, diminishes with the altitude. Near the horizon—say within 10 degrees of it—its effect is very uncertain, and observations of objects in that position are therefore unreliable. At an altitude of 45° the refraction is about 1′. As it varies with the temperature and atmospheric pressure the barometer and the thermometer must be read if very exact results are required.

The corrections for refraction and parallax are not to be found in the Nautical Almanac, but are given in all sets of mathematical tables. The N. A., as a rule, gives only variable quantities — such as declination, right ascension, equation of time, etc. It is rather a bulky volume, but the portions of it in general use by the practical surveyor could be comprised in a small pamphlet. The most useful are the sun's declination and right ascension, the equation of time, the sun's semi-diameter, and the sidereal time of mean noon—all given for every day in the year; the declinations and right ascensions of the principal fixed stars, taken in regular order according to their right ascensions; and the tables for converting intervals of mean time into sidereal time and *vice versa*. To these may be added tables of moon-culminating stars, and tables for finding the latitude from the altitude of the pole star when off the meridian.

Specimens of the two first pages of the quantities given for each month in the Nautical Almanac, and of the data for fixed stars, are reprinted below. All the quantities are given for noon at Greenwich on the day in question. They must, therefore, be corrected by a proportion for any other hour or longitude. Thus, when it is noon at a place in 90° west longitude, or six hours west of Greenwich, it is 6 p.m. at the latter. Therefore, if an observation were taken at the western station at noon the quantities required would have to be corrected for their change in six hours.

Owing to the earth's uniform revolution round its axis a change in longitude affects alike mean time and sidereal time. Thus, if the mean time at Greenwich was 3 p.m., and the sidereal time 11h., they would be 9 a.m. and 5h. respectively at a place in longitude 90° west.

In the Nautical Almanac the day is supposed to commence at noon and to last for 24 hours. Thus 9 a.m. on the 2nd of January, ordinary civil reckoning, is 21h. of the 1st of January. This astronomical method of reckoning mean time must not be confounded with sidereal time, which is quite a different thing.

The data for the first of each month are given also at the end of the preceding month. Thus, we find 32 days given in January, the one shown as the 32d being really the 1st of February. This is for convenience in interpolating.

JUNE, 1880.

AT APPARENT NOON.

Day of the Week.	Day of the Month.	THE SUN'S Apparent Right Ascension.	THE SUN'S Var. in 1 hour.	THE SUN'S Apparent Declination.	THE SUN'S Var. in 1 hour.	Sidereal time of the Semi-diameter passing the Meridian.	Equation of Time to be ~~subt. from~~ added to Apparent Time.	Var. in 1 hour.
		h m s	s	° ′ ″	″	m s	m s	s
Tues.	1	4 39 0·71	10·241	N.22 8 59·6	19·79	1 8·43	2 21·83	0.384
Wed.	2	4 43 6·71	10·258	22 16 42·9	18·82	1 8·49	2 12·42	0·400
Thur.	3	4 47 13·10	10·274	22 24 2·9	17·85	1 8·54	2 2·61	0·416
Frid.	4	4 51 19·87	10·289	22 30 59·4	16·86	1 8·59	1 52·43	0·431
Sat.	5	4 55 26·99	10·304	22 37 32·3	15·87	1 8·63	1 41·90	0·446
Sun.	6	4 59 34·44	10·317	22 43 41·3	14·88	1 8·68	1 31·03	0·459
Mon.	7	5 3 42·20	10·329	22 49 26·5	13·88	1 8·72	1 19·86	0·471
Tues.	8	5 7 50·24	10·340	22 54 47·6	12·88	1 8·75	1 8·41	0·483
Wed.	9	5 11 58·54	10·351	22 59 44·6	11·87	1 8·79	0 56·70	0.493
Thur.	10	5 16 7·08	10·360	23 4 17·3	10·86	1 8·82	0 44·75	0·502
Frid.	11	5 20 15·82	10·368	23 8 25·7	9·84	1 8·85	0 32·60	0·510
Sat.	12	5 24 24·74	10·375	23 12 9·6	8·82	1 8·87	0 20·27	0·517
Sun.	13	5 28 33·81	10·381	23 15 29·0	7·79	1 8·89	0 7·79	0·523
Mon.	14	5 32 43·02	10·386	23 18 23·7	6·77	1 8·91	0 4·83	0·528
Tues.	15	5 36 52·34	10·380	23 20 53·8	5·74	1 8·93	0 17·35	0·532

JUNE, 1880.

AT MEAN NOON.

Day of the Week.	Day of the Month.	THE SUN'S *Apparent* Right Ascension.	THE SUN'S *Apparent* Declination.	THE SUN'S Semi-diameter.	Equation of Time, *to be added to* / *subt. from* *Mean Time.*	Sidereal Time.
		h m s	° ′ ″	′ ″	m s	h m s
Tues.	1	4 39 1·11	N.22 9 0·4	15 48·1	2 21·82	4 41 22·93
Wed.	2	4 43 7·09	22 16 43·6	15 47·9	2 12·40	4 45 19·49
Thur.	3	4 47 13·45	22 24 3·5	15 47·8	2 2·59	4 49 16·05
Frid.	4	4 51 20·19	22 30 59·9	15 47·7	1 52·41	4 53 12·60
Sat.	5	4 55 27·28	22 37 32·7	15 47·5	1 41·88	4 57 9·16
Sun.	6	4 59 34·70	22 43 41·7	15 47·4	1 31·02	5 1 5·72
Mon.	7	5 3 42·43	22 49 26·8	15 47·3	1 19·85	5 5 2·28
Tues.	8	5 7 50·44	22 54 47·9	15 47·2	1 8·40	5 8 58·83
Wed.	9	5 11 58·71	22 59 44·8	15 47·1	0 56·69	5 12 55·39
Thur.	10	5 16 7·21	23 4 17·5	15 47·0	0 44·75	5 16 51·95
Frid.	11	5 20 15·91	23 8 25·8	15 46·9	0 32·60	5 20 48·51
Sat.	12	5 24 24·80	23 12 9·7	15 46·8	0 20·27	5 24 45·07
Sun.	13	5 28 33·84	23 15 29·0	15 46·7	0 7·79	5 28 41·62
Mon.	14	5 32 43·01	23 18 23·7	15 46·7	0 4·83	5 32 38·18
Tues.	15	5 36 52·29	23 20 53·9	15 46·6	0 17·55	5 36 34·74

APPARENT PLACES OF STARS, 1880.

AT UPPER TRANSIT AT GREENWICH.

Month and Day.	α Andromedæ R.A.		α Andromedæ Dec. N.		γ Pegasi. (*Algenib,*) R.A.		γ Pegasi. Dec. N.		ι Ceti. R.A.		ι Ceti. Dec. S		β Hydri. R.A.		β Hydri. Dec. S.	
	h m		° ′		h m		° ′		h m		° ′		h m		° ′	
	0 2		28 25		0 7		14 31		0 13		9 28		0 19		77 55	
	s		″		s		″		s		″		s		″	
Jan. 1	11·88	13	55·3	9	4·24	11	8·6	9	19·61	11	80·5	6	26·30	93	62·7	12
11	11·75	13	54·4	12	4·13	11	7·7	9	19·50	11	81·1	3	25·37	86	61·5	17
21	11·62	11	53·2	14	4·02	9	6·8	11	19·39	9	81·4	2	24·51	78	59·8	22
31	11·51		51·8		3·93		5·7		19·30		81·6		23·73		57·6	

To revert to the subject of sidereal time: Since the sidereal clock stands at zero or 24h. at the instant the 1st point of Aries is on the meridian, and as the clock keeps time with the stars in th~~is~~eir apparent diurnal revolution round the earth, it follows that when any particular star is on the meridian its right ascension is the sidereal time of the instant. Thus, if the stars R. A. were 6h. the clock should show that time at the instant of the stars transit, and its error may be ascertained by mounting a telescope so as to move only in the plane of the meridian, and noting the instant of transit. If we want to find the *mean* time of a star's transit we have only to convert the star's R. A. into the corresponding mean time of the instant, in the manner to be presently explained. Conversely, a star's transit gives us the sidereal time of the instant, and hence the true mean time.

The celestial globe is of great use in studying astronomy. It is a model of the great sphere supposed to be viewed from outside. The positions of the stars on it are the points where straight lines, joining them with the earth, would intersect it. The equator and ecliptic—the latter being the sun's annual path through the stars—are marked on it, as also the sun's place in the ecliptic for every five days. The axis on which it turns is that of the poles. The metal ring passing through the latter represents the meridian, and the flat horizontal ring the plane of the rational horizon.

One of the chief uses of the globe is to show the position of the stars at any instant with regard to the spectator. To do this we raise the pole by means of the graduation on the meridian so as to give it an altitude above the horizon equal to the latitude of the place, and bring the sun's place in the ecliptic for the day to the meridian. The half of the globe above the horizon will now roughly represent the position of the visible hemisphere at noon. To find the position of the sphere at any

other hour it is only necessary to turn the globe through an hour angle equivalent to the number of hours from noon. Thus, if we wanted to find out the visible positions of the stars at 8 P. M. we should have to revolve the globe westwards through an angle of 120°. Conversely, we can find the name of any constellation or star by noting its position in the sky and the hour, and setting the globe accordingly.

The great circle passing through the zenith and the east and west points of the horizon, and therefore at right angles to the meridian, is called the "prime vertical."

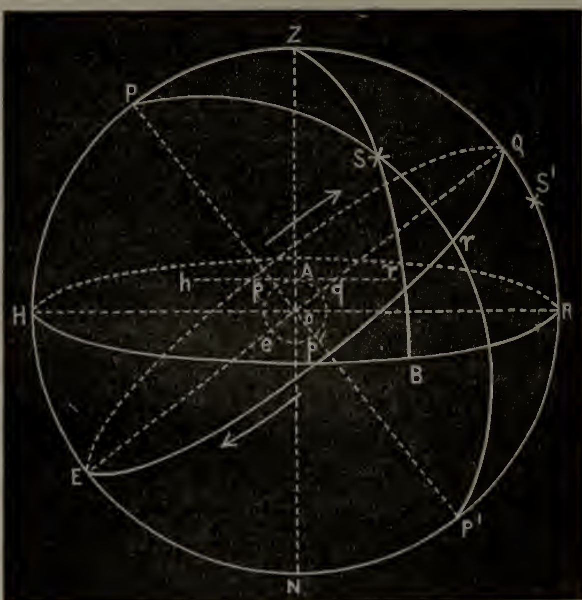


Fig. 3.

In Figure 3 the small circle at the centre represents the earth, and the large circle the great sphere. Strictly speaking the former should be a mere point in comparison to the latter, and the points on the great sphere would ap-

pear at A, the spectators position, in the same places as if viewed from O, the earth's centre. $h\,r$ is a tangent to the earth's surface at A, and therefore lies in the plane of the observer's horizon. H B R is the plane of the rational horizon, and $p\,p^1$ the earth's polar axis, meeting the great sphere in the points P P^1. Z is the zenith. N the nadir, or point on the sphere diametrically opposite it. The plane of the paper represents the plane of the observers meridian, and H, R, are the north and south points of the horizon, $e\,q$ is the equator, E γ Q the equinoctial, γ the first point of Aries, and P γ P^1 the initial declination circle passing through it, from which all right ascensions are reckoned. S is a star situated on it, and S^1 another star on the observer's meridian. Z S B is a portion of a great circle, passing through the zenith and S and meeting the horizon at B. Z B is of course 90°. The arrows represent the apparent motion of the great sphere with respect to the earth.

The arc A♈, or the angle A O q, is the latitude of the point A, and A O q=Z O Q, which is the zenith distance of the point where the equator cuts the meridian. Also, Z O Q=90°—P O Z=P O H; or the altitude of the visible pole above the horizon is the latitude of the place. It should be noticed that the whole of the hemisphere above the plane H B R is visible to the observer at A.

The right ascension of the star S is zero, its declination (north) S γ, hour angle S P Z, its altitude S B, zenith distance S Z and azimuth S Z R. The star S^1 has R. A. γ Q, declination (south) Q S′, hour angle nil, altitude S^1 R, zenith distance, S^1 Z, and azimuth zero. The sidereal time of the instant is γ P Q, or the arc γ Q. It is therefore the same as the R. A. of the meridian. The triangle P Z S is called the "astronomical triangle." It should be noted that in all calculations if north declination is reckoned positive, south declination must be counted negative, and *vice versa*.

CHAPTER III.

Uses of practical astronomy to the surveyor. Instruments employed in the field. Methods of using them. Taking altitudes. Problems relating to time.

The principal uses of practical astronomy to the surveyor are that it enables him to ascertain his latitude, longitude, local mean time, and the azimuth of any given line; the latter of course giving him the true north and south line and the variation of the compass. In fact the only check he has on his work as regards direction when running a long straight line across country is by determining its true azimuth from time to time, allowing (as will be explained hereafter) for the convergence of meridians. The instruments usually employed are the transit theodolite, sextant or reflecting circle with artificial horizon, solar compass, portable transit telescope, and zenith telescope. To these must be added a watch or chronometer keeping mean time, a sidereal time chronometer (this is not, however, absolutely essential), the Nautical Almanac for the year, and a set of mathematical tables. With the sextant or reflecting circle we can measure altitudes and work out all problems depending on them alone, and also lunar distances. The transit theodolite may be used for altitudes, and also gives azimuths. The solar compass is a contrivance for finding, mechanically, the latitude,

meridian line, and sun's hour angle. The zenith telescope gives the latitude with great exactness, and is particularly suited to the work of laying down a parallel of latitude. The transit telescope enables us to determine the mean and sidereal time, latitude, and longitude. The transit theodolite answers the same purpose, but is not so delicate an instrument. It is, however, of almost universal application, and nearly every problem of practical field astronomy may be worked out by its means alone if the observer has a fairly good ordinary watch. The sextant has been called a portable observatory; but in the writer's opinion the term is more applicable to the last named instrument. The sextant is not so easy to manage and only measures angles up to about 116,° so that 58° is practically the greatest attitude that can be taken with it when the artificial horizon has to be used. The latter, as generaliy made, is disturbed by the least wind, and then gives a blurred reflection, making the observation nearly worthless. There is little use in having the arc graduated to read to within a few seconds if the contact of the images cannot be made with certainty to within a minute or two.

All observations taken with the transit theodolite should, if the nature of the case admits of it, be repeated in reversed positions of the telescope and horizontal plate, and the mean of the readings taken, as we thereby get rid of the effects of collimation, index, and other instrumental errors. Thus, for an altitude, the plate having been levelled, the vertical arc set at zero, and the bubble of the telescope level brought to the middle by the twin screws, the verticality of the axis is tested by turning the upper plate in azimuth 180°, and seeing if the bubble is still in the centre. If it is not it is corrected, half by the lower plate screws, half by the twin screws, and the operation repeated till the bubble remains in the centre in every position. The altitude is then taken, the telescope

turned over, the upper plate turned round, and the altitude again read. In each case both verniers should be read.

The first step after taking an altitude with either sextant or theodolite is to correct it for index error, if there is any. The following lists give the corrections to be applied in each case to an altitude of the sun's upper or lower limb to obtain that of his centre :

THEODOLITE.	SEXTANT. Altitude above water horizon.	SEXTANT. Double Altitude with artificial horizon.
Index error.	Index Error.	Index Error.
	Dip of Horizon.	Divison by 2.
Refraction.	Refraction.	Refraction.
Parallax.	Parallax.	Parallax.
Semi-diameter.	Semi-diameter.	Semi-diameter.

The semi-diameter has to be added if the lower limb is observed, and *vice versa*. When taking an altitude for time with the artificial horizon the easiest way to get the correct instant of contact is to bring the two images into such a position that they overlap a little while receding from each other. At the instant they just touch the observer calls "stop," the assistant notes the exact watch time, and the vernier is then read. This plan necessitates observing the lower limb in the forenoon and the upper in the afternoon. The dip depends on the height of the instrument above the water, and, like the refraction and parallax, is to be found in the mathematical tables.

In the case of a meridian altitude for latitude the sun or star, after rising to its greatest height, appears for a short time to move horizontally. When this is the case the altitude may be read off.

Fixed stars require, of course, no correction for parallax or semi-diameter. As the refraction tables require a correction for temperature and atmospheric pressure the height of the thermometer and barometer should be noted.

If an altitude has to be taken with the sextant and artificial horizon, and the sun is too high in the heavens for the instrument, a suitable star must be observed instead.

In surveying operations the latitude is generally known approximately. This gives the approximate altitude for a meridian observation; for the altitude of the intersection of the meridian and equator being 90° minus the latitude, we have only to add to or subtract from this quantity the objects declination, and we have the altitude.

Principal CAUSE OF THE EQUATION OF TIME.

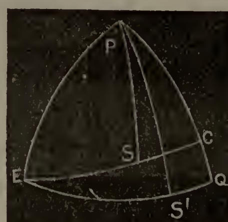


Fig. 4.

In Figure 4 P is the pole, E C a portion of the ecliptic, and E Q a portion of the equator; each being equal to 90°. C and Q are on the same meridian, and P Q is also a quadrant. Now, let S be the sun, and suppose it to move at a uniform rate from E to C. Let S^1 be an imaginary sun (called the "mean" sun) moving in the equator at the same rate as the real sun. Now, let the two suns start together from E, and after a certain interval let their position be as shown in the figure. Since they move at the same rate, E S will be equal to E S^1, but as a consequence the meridians P S and P S^1 will not coincide, S^1 having got ahead of S. The angle S P S^1 formed by the two meridians is the equation of time. As the two suns must arrive simultaneously at C and Q it is evident that, though S^1 gains on S at first, it will, after a certain point, cease to gain and lose instead. *

Since the equation of time—in other words the difference between apparent and mean solar time—is continually changing, if we want to find from the Almanac the mean time corresponding to apparent time at any

* The other main cause of the equation of time is the earth's elliptic orbit, & variable instead of uniform motion

particular instant and longitude, we must allow for the change in the equation that has taken place since noon at Greenwich. For instance; suppose we had to find the mean time corresponding to three hours P.M. apparent time on the 22nd April, 1882 at a place in longitude 6h. west. By the N. A. the equation of time at apparent noon that day at Greenwich was 1m. 34s. 43, to be subtracted from apparent time and increasing, the variation per hour 0.s.496. At 3 P.M. at the place it would be 9 P.M. at Greenwich. $9 \times 0s.496 = 4s.464$. The corrected equation of time is 1m. 38s.89, and the true mean time 2h. 58m. 21s.11 P.M.

GIVEN THE SIDEREAL TIME AT A CERTAIN INSTANT TO FIND THE MEAN TIME.

Here we have given the right ascension of the declination circle of the great sphere that is on the meridian at the instant, or—which is the same thing—the time that a sidereal clock would show. Now the Nautical Almanac gives the sidereal time of mean noon at Greenwich, which has to be corrected for longitude. These two data give us the interval in sidereal time that has elapsed since mean noon, and this, converted into mean time units, will be the mean time.

Ex. Find the mean time corresponding to 14 hours sidereal time at Kingston on the 28th April, 1882.

We find from the N. A.

Sidereal time of mean noon at Greenwich..........	2h. 25m. 25s·33
Correction for longitude............................	50 ·26
Sidereal time of mean noon at Kingston..........	2h. 26m. 15s·59
Sidereal time of the instant........................	14h. 0m. 0s
Difference, or interval of sidereal time that has elapsed since mean noon	11h. 33m. 44s·41
Which, converted into mean time, is..............	11h. 31m. 50s·75 P. M.

The conversion of sidereal into mean time units, and *vice versa*, is obtained from tables at the end of the Nautical Almanac.

If the sidereal time of mean noon is greater than the sidereal time given we shall obtain the interval before mean noon. Thus, if on the same date as above we wanted to find the mean time corresponding to sidereal time one hour we should proceed as follows:

Sidereal time of mean noon	2h.	26m.	15s·69
" " of the instant	1	0	0
Sidereal interval before mean noon................	1	26	15·69
Which in mean time units is......................	1	26	1·56
Subtracting this from 12h........................	12	0	0
We have mean time........................	10h.	33m.	58s·44 A.M.

It is sometimes convenient to add 24 hours to the given sidereal time to make the subtraction possible. Thus, if the sidereal time were 1h., and the sidereal time of mean noon 23h., we should have the interval elapsed since mean noon 25—23, or 2 hours, which is 1h. 59m. 40s.3 P.M., mean time.

TO FIND THE MEAN TIME AT WHICH A GIVEN STAR WILL BE ON THE MERIDIAN.

This is only an application of the preceding problem. For the Almanac gives us the star's right ascension, which is the same thing as the sidereal time of its culmination, and we have merely to find the mean time corresponding to it.

GIVEN THE LOCAL MEAN TIME AT ANY INSTANT TO FIND THE SIDEREAL TIME.

Here we must convert the interval in mean time that has elapsed since the preceding noon into sidereal units, and add to it the sidereal time of mean noon.

Ex. Find the sidereal time at 9 A.M., on the 29th of April, 1882, at Kingston, Canada.

Here we have, as before:

Sidereal time of mean noon on the 28th.................	2h.	26m.	15s·69
Add 21 hours of mean time in sidereal units, or..........	21	3	26·98
Sidereal time	23h.	29m.	42s·67

If this process makes the result more than 24 hours that number must, of course, be subtracted from it. Thus, if we got 25 hours the sidereal clock would show 1h. If the sidereal time of mean noon is greater than the interval in sidereal units we add 24 hours to the latter to make the subtraction possible.

The correction on account of longitude for the sidereal time of mean noon is constant for any particular place or meridian

The subject of sidereal time may be thus illustrated:

In Fig. 5 let the small circle represent the earth, and the large circle the equator of the great sphere viewed from the north, the plane of the paper being the plane of the equator. Let P be the pole, A a point on the earth's surface, and P A the meridian of A. γ is the first point of Aries, S and S¹ two stars, and *s* and *s*¹ the points where their declination circles meet the equator. Now the arc $\gamma\ s^1$ (or the angle γ P s^1) is the right ascension of S_1, and the arc $\gamma\ s^1\ s$ that of S. Now suppose the earth (and therefore the meridian P A *m*) to remain fixed, while the outer circle and stars revolve around it in the direction of the arrow; and at the instant that it is mean noon on a certain day at A let the position of the great sphere be as shown in the figure. The arc $\gamma\ s^1\ m$ will be the sidereal time of mean noon for that day at A. The star S will be on the meridian at an interval of sidereal time after mean noon corresponding to *s m*, while the star S¹ has passed the meridian by an interval corresponding to $m\ s_1$, and by reducing these intervals to their equivalents in mean time we shall have the mean times of their transits. For

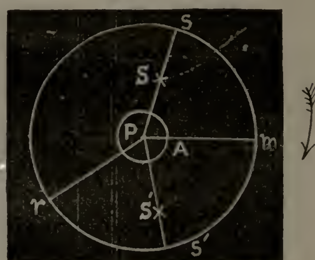


Fig. 5.

instance, suppose we had to find at what time the pole star would be at its upper transit on a day when the sidereal time of mean noon was 21h. 30m., the right ascension of the star being taken as 1h. 15m. Now the state of things at noon would be as shown in Fig. 6. The star would have passed the meridian by an interval of 21h. 30m.—1h. 15m., or 20h. 15m. (sidereal) and would therefore be on the meridian at 3h. 45m. sidereal, or 3h. 44m. 23s. mean time after noon.

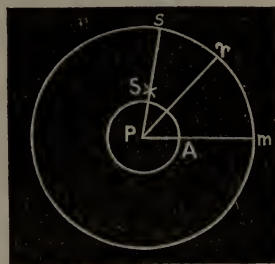


Fig. 6.

Sidereal time is usually found by calculating the hour angle of a star from its observed altitude. This, added to the star's right ascension if the hour angle is west, or subtracted from it if east, gives the sidereal time. From this the mean time can be obtained, if required. The watch time at which the altitude is observed must, of course, be noted.

TO FIND THE HOUR ANGLE OF A GIVEN STAR AT A GIVEN TIME AT A GIVEN MERIDIAN.

Here we must find the local sidereal time of the given instant and take the star's right ascension from the Almanac. The difference between these two quantities will be the star's hour angle, which will be east if the star's R. A. is greater than the sidereal time, and west if the contrary is the case.

TO FIND THE MEAN TIME BY EQUAL ALTITUDES OF A FIXED STAR.

Fixed stars are employed for this purpose in preference to the sun or planets because of the change in declination of the latter. A star should be chosen which describes a sufficiently high arc in the sky. Two or three hours before its culmination its altitude is taken with the

sextant or theodolite, the exact watch time noted, and the instrument left clamped at that altitude. Some hours later, when the star has nearly come down to the same altitude, the observer looks out for it (keeping the instrument still clamped) till it enters the field of view of the telescope, and waits till it has exactly the same altitude as before, when he again notes the watch time. The mean of the times of equal altitude will give the watch time of the star's culmination, which should be the same as the mean time (previously calculated), corresponding to the star's right ascension, the latter being the sidereal time of the culmination. If they are not the same the difference will be the watch error.

If the theodolite is used for this observation the vertical arc only is kept clamped. When the star has nearly come down again to the ~~second~~ same altitude the horizontal arc is clamped and its slow motion screw used.

TO FIND THE LOCAL MEAN TIME BY AN OBSERVED ALTITUDE OF A HEAVENLY BODY.

For this problem we must know the latitude of the place, and, if the sun is the object observed, we must also know the mean time approximately in order to correct its declination.

The altitude should be taken when the heavenly body is rapidly rising or falling—that is, as a rule, when it is about three hours from the meridian, and the nearer to the prime vertical the better.

If we take P as the pole, Z the zenith, and S the heavenly body (Fig. 7), PZS will be a spherical triangle in which PZ is the complement of the latitude, PS the polar distance of the object observed, and ZS the complement of the altitude. The three sides being given we can find the three angles from the usual formulæ. In the present instance we

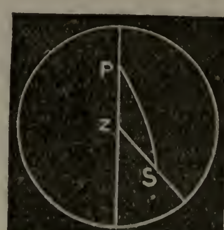


Fig 7.

want P, which is the hour angle of the object.

A convenient formula is

$$\operatorname{Sin}^2 \frac{P}{2} = \frac{\sin(s-PS)\sin(s-PZ)}{\sin PS \sin PZ}$$

where s is $\dfrac{PZ+PS+ZS}{2}$

The watch time is noted by an assistant at the instant the altitude is taken. The usual corrections are applied to the altitude, and the angle P having been worked out is divided by 15. This gives us the hour angle of the body. In the case of the sun it will be the apparent solar time, and by adding or subtracting the corrected equation of time we shall get the true mean time of the instant of the observation.

If the body is a fixed star or planet, then, from its known right ascension we subtract the hour angle if it is east, or add it if it is west. This gives the sidereal time of the instant, from which the mean time can be inferred if required. The spherical triangle PZS is known as the "astronomical triangle."

Instead of taking a single altitude two or three may be taken in quick succession and the time of each noted.

The mean of the altitudes is then taken as a single altitude to correspond to the mean of the times. If the transit theodolite is used two altitudes should be taken in reversed positions of the telescope and horizontal plate so as to correct instrumental errors. Still greater accuracy is obtained by observing both an east and a west star and taking the mean of the results, as errors of observation, refraction, and of the instrument, will be got rid of in a great measure.

EXAMPLE OF WORKING OUT A SEXTANT DOUBLE ALTITUDE OF THE SUN'S LOWER LIMB FOR LOCAL TIME.

17th April, 1882.—Lat. 44° 13′ 40″ N., Long. 5h. 5m. 50s. W. Sun's semi-diameter, 15′ 57″. Declination 10° 40′ 0″ N. Watch time of observation, 3h. 37m. 15s. P.M. Equation of Time, 0m. 34s., to be subtracted from apparent time. Index Error, 5′ 30″, *on*—

Double altitude	64°	4′	0″
Index error		5	30
	2)63	58	30
	31	59	15
Semi-diameter		15	57
	32	15	12
Refraction and parallax ...		1	23
True altitude of sun's centre	32	13	49
	90		

These corrections should be applied before semi-diameter

	90	0	0	57	46	11=ZS
Declination	10	40	0	79	20	0 =PS
				45	46	20=PZ
	79	20	0			
				2)182	52	31
	90	0	0			
Latitude...	44	13	40	91	26	15=*s*
				79	20	0
	45	46	20			
	91	26	15	12	6	15=*s*—PS
	45	39	55.........	45	39	55=*s*—PZ

Log sin (*s*—PZ)= 9·8544700
" (*s*—PS)= 9.3215800
Log cosec PZ =10.1447400
" cosec PS =10.0075700

2)39·3283600

19·6641800=log sin 27° ~~19′~~ 29′ +10
2

54 ~~58~~ 58=P
=3h. 39m. 52s.

Equation of time =	— 0		34
True mean time =3	~~38~~ 39		18
Watch time 3	37		15
Watch slow		2m.	3s.

EXAMPLE OF CORRECTING A MEAN TIME WATCH BY A SINGLE ALTITUDE OF A STAR.

On a certain date at a place in 5h. 30m. west longitude the altitude of a star was taken at 9 P.M. by the watch, and the hour angle when worked out proved to be 2h. 30m. 17s. west.

To find the watch error—

The star's right ascension was	4h.	29m.	4s.
Add hour angle............	2	30	17
Sidereal time of the instant=	6	59	21
Add...........................	24		
	30	59	21
Subtract the sidereal time of mean noon, corrected for longitude	22	0	9
Sidereal interval since mean noon	8	59	12

= 8h. 57m. 44s. in mean time.

And the watch was 2m. 16s. fast.

It should be noted that if the declination of a heavenly body is north it must be subtracted from 90° to get its polar distance. If south the declination must be added to 90°.

The following formula is a very convenient one, as the altitude and latitude are employed instead of their complements.

$$\text{Sin}^2\frac{P}{2}=\frac{\cos s \sin (s-a)}{\cos \lambda \sin PS}$$;where a is the altitude, λ the latitude, PS the polar distance, and $s=\frac{a+\lambda+PS}{2}$

TO FIND THE TIME BY A MERIDIAN TRANSIT OF A HEAVENLY BODY.

This is a very simple method when the direction of the true north is known, as when running north and south lines on a survey. The theodolite is set up on the line,

the telescope directed on a distant point or mark on it, and the horizontal plate clamped. The telescope will now, if moved in altitude, keep in the plane of the meridian, provided the instrument is in adjustment; and the instant of transit of any object across the vertical wire or intersection being noted, we can deduce the true time.

As the altitude of an object at transit is equal to the altitude of the intersection of the meridian and equator plus or minus the declination of the object, we have the equation

$$\text{Altitude} = 90° - \text{latitude} \pm \text{declination}.$$

and can set the telescope beforehand at the required altitude. If the latter is more than 50° a diagonal eye piece is necessary with most instruments. In the case of the sun we may either take the mean of the observed instants of transit of the east and west limbs, or take the transit of one limb and add or subtract the time required for his semi-diameter to pass the meridian (which we obtain from the Almanac). We now have the *watch* time of transit of the sun's centre, which takes place at apparent noon, and have only to find the *true* mean time of apparent noon by adding to or subtracting from the latter the equation of time (corrected for longitude), when the difference will give us the watch error.

Example—At Kingston, on the 2nd of May, 1882, the transit of the sun's west limb was observed at 11h. 55m. A. M. What was the watch error?

Here we have

Watch time of transit of limb..............	11h. 55m. 0s.
Time of the semi-diameter passing the meridian.................................	1m. 6s.
Watch time of transit of sun's centre.....	11h. 56m. 6s.

The equation of time, corrected for longitude, was 3m. 11.5s, to be subtracted from apparent time.

Apparent time of transit of sun's centre	12h. 0m. 0s.
Equation of time..................	3m. 11.5s.
True mean time of transit..................	11h. 56m. 48.5s.
Watch time of transit........................	11h. 56m. 6.0s.
Watch slow..........................	42.5s.

In the case of a star its right ascension is the sidereal time of the instant of transit, and by working out the corresponding mean time in the usual way we get the watch error.

If a planet is to be observed we take its right ascension from the part of the Almanac which gives its position *at transit at Greenwich*, correcting it for longitude in the way directed in the explanations at the end.

For observing objects at night the theodolite ought to have an illuminating apparatus to light up the wires. The plan of throwing the light of a lantern through the object glass is objectionable if it can be avoided.

It should be noted that the nearer the observed object is to the zenith the less will be the effect of any error in the direction of the north and south line, and the greater that of one of the telescope pivots being higher than the other.

With small instruments, objects near the equator, from moving the fastest, are preferable, while those nearest the pole are the worst.

CHAPTER IV.

METHODS OF FINDING THE LATITUDE, LONGITUDE, AND MERIDIAN.

TO FIND THE LATITUDE BY THE MERIDIAN ALTITUDE OF THE SUN OR A STAR.

The altitude may be taken with the theodolite or sextant. The approximate direction of the meridian should be known beforehand. If the theodolite is used the instrument is levelled, and the telescope directed on the object a little before it attains its greatest height. The horizontal wire is then made to touch the object (the lower limb if the sun is observed) and is kept in contact with it as it rises by means of the slow-motion screw of the vertical arc, the telescope being moved laterally as required. When the object has attained its greatest altitude it will remain for a short time in contact with the wire, when the vertical arc is read off. The telescope is then at once turned over, the upper plate reversed, and the altitude again read. The mean of the two readings will be the apparent altitude of the object.

When using the sextant and artificial horizon we bring the two images into contact and keep them so by the slow motion screw till they cease to separate, when the vernier is read off.

The usual corrections having been applied, and the true meridian altitude of the sun's centre or star thus obtained, the latitude is found as follows:

1st case. Let the object observed culminate at the opposite side of the zenith to the visible pole.

In Fig. 8 let A be the observer's position, p A q p^1 e a section of the earth passing through A, and the poles (p p^1) and therefore in the plane of the meridian.

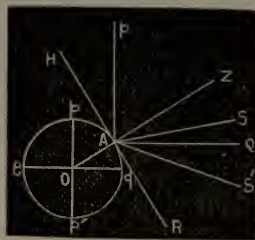


Fig. 8

Let O be the earth's centre and let e O q be perpendicular to p p^1; then e and q will will be the intersections of the equator with the meridian. Draw H A R touching the earth's surface at A and also in the plane of the meridian. H R will be in the plane of the horizon and will lie due north and south. Let S be the object observed and let its declination be north. Draw A P parallel to p p^1 and A Q parallel to e q. A P will be the direction of the pole of the great sphere, and A Q that of the intersection of the meridian and equinoctial. Join O A and produce it to *Z* the zenith. A *Z* is at right angles to H R, and P A Q is also a right angle. S A R is the measured altitude of the object and S A Q its declination. Now the latitude of A is the arc A q or the angle A O q. But A O q=*Z* A Q, which is the declination of the zenith. Also, *Z* A Q=90°−P A *Z*=P A H, which is the altitude of the visible pole. Hence we have:

Latitude=*Z*AQ=90°−QAR=90°—(S A R−S A Q)
=90°−altitude+declination.

If the object had south declination, as S^1, we should have (since S^1 A R is its altitude and S^1 A Q its declination) Latitude=*Z* A Q=90°−Q A R=90°—(S^1 A R + Q A S^1)=90°—altitude−declination.

In working out latitudes it is always best to draw a figure. In the one given p is the north and p^1 the south pole.

2nd case. If the object culminates between the zenith and the visible pole, as at S in figure 9, its altitude will be S A H, and we shall have:

$$\begin{aligned}\text{Latitude} &= \text{P A H} = \text{S A H} - \text{S A P} \\ &= \text{S A H} - (90^\circ - \text{S A Q}) \\ &= \text{altitude} + \text{declination} - 90^\circ.\end{aligned}$$

If the object is a star which never sets, but describes a diurnal circle round the pole, it will cross the meridian twice in the 24 hours, and we may take its altitude at what is known as its lower transit, as at S^1, Fig. 9. Here we have:

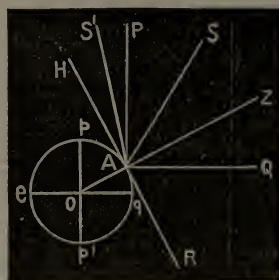


Fig 9.

Latitude $=$ P A H $=$ S^1 A H $+$ S^1 A P $=$ altitude $+ 90^\circ -$ declination. Therefore, in the case of such a star we have:

Latitude $=$ star's altitude $\pm$ star's polar distance; the positive sign being taken if the star is observed below the pole, and *vice versa*.

Case 2 can only apply to the sun when A is within the tropics. In many books on astronomy the formulœ of case I are made to apply to the sun in every situation, whereas they manifestly fail when he culminates between the zenith and the visible pole.

In the Nautical Almanac is given a very simple method of finding the latitude from an altitude of the pole star taken at any point of its diurnal circle round the pole. The time of the observation has to be noted and the corresponding sidereal time calculated.

LONGITUDE.

Longitude cannot, like latitude, be measured absolutely,

as it has no natural zero or origin, and we have to assume an initial meridian arbitrarily, the English adopting that of Greenwich. But the difference of longitude of two places can always be found. The simplest method of doing this is by comparing the local time at the two places for the same instant. This is done by signal of some kind or other, such as flashing the sun's rays from station to station, or by the electric telegraph.

Since the earth revolves through 360 degrees of hour angle in 24 hours it will pass through 15 degrees in 1 hour. That is, when it is one o'clock in the afternoon at a certain station it will be 2 o'clock at a station 15 degrees to the east of it. Fifteen minutes and seconds of longitude in arc will be equivalent to one minute and second of time respectively. This applies to sidereal as well as to mean time. That is, if the sidereal clock showed 1 hour at one place it would show 2 hours at the other. (This ~~apparent paradox~~ is worth thinking out, for it is often a puzzle to beginners.) Therefore, if at a preconcerted signal the observers at two stations note the exact local time, either mean or sidereal, the difference of the two will give the difference of longitude. Ordinary watches may be used if their exact rate and their error at any given instant are known. The best way, however, is by telegraphing star transits. If the eastern observer signals at the instant that a certain star is on the meridian, and the western observer notes the time of the signal by his sidereal chronometer and afterwards takes the time of the star's transit at his own station, the interval of time between the two transits, allowing for the clock's rate, will evidently give the difference of longitude. If the eastern observer notes the time of the transit, and the western signals the transit at his station, the same result will be obtained, and by taking the mean any time lost in the transmission of the signals will be corrected; for it is evident that in the first case the time lost will make the difference of longitude too

small, and in the second case it will make it too large. By means of certain contrivances it is possible to register the instant of a transit to a small fraction of a second, and if a number of observations are taken the mean of the results will be very near the truth.

Since an error of one second in the time will throw the longitude out by about 360 yards in latitude 45° it is evident that for surveying purposes great care must be taken to insure accuracy. When the local times are compared by flashing signals a large number of observations should be made and the results compared.

The subject of longitude will be more fully gone into hereafter. It may be mentioned here that sailors obtain their longitude by finding the ship's local mean time by an altitude of the sun when the latter is about three hours from the meridian, and comparing it with a chronometer keeping Greenwich mean time, the latter being noted at the instant the altitude is taken. It seems almost needless to remark that when using chronometers the correction for rate must always be applied.

METHODS OF FINDING THE MERIDIAN.

TO FIND THE AZIMUTH OF A HEAVENLY BODY FROM ITS OBSERVED ALTITUDE.

This is a very similar problem to that of finding the hour angle from an altitude; the only difference being that, instead of finding the angle P of the triangle P *Z* S, we have to find the angle *Z*. We have, as before, the three sides of the triangle given, and may therefore use the formula

$$\text{Sin}^2 \frac{Z}{2} = \frac{\text{Sin } (s - Z\text{ S}) \text{ Sin } (s - Z\text{ P})}{\text{Sin } Z\text{ S Sin } Z\text{ P}}$$

Another formula that may be employed is:

$$\text{Cos}^2 \frac{Z}{2} = \text{Cos } s \text{ Cos } (s - \text{S P}) \text{ Sec } \lambda \text{ Sec } \alpha$$

where α is the altitude of the object, S P its polar distance, λ the latitude, and $s = \dfrac{\alpha + \lambda + \text{P S}}{2}$

This formula is rather the most convenient of the two, as it entails less subtraction than the other.

This problem (which is one of great importance to surveyors) is usually solved by means of ~~requires, of course,~~ an alt.-azimuth instrument, such as a transit theodolite. The astronomical bearing of any line Z A (Fig. 10) can be found in this way by directing the telescope on the line or mark, taking the horizontal plate reading, and then turning it on the heavenly body and taking its altitude and the horizontal plate reading. It is better to repeat the observation in reversed positions of the instrument and take the mean of the two sets of readings. The difference of the horizontal readings on the line and the heavenly body gives the angle A Z S, and the triangle P Z S when solved gives the angle P Z S, whence we have A Z P the required bearing, and therefore Z P the direction of the meridian.

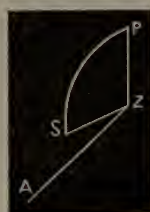


Fig 10.

In taking an alt-azimuth of the sun, if we take a single altitude only we must add or subtract the semi-diameter to get the altitude of the centre. If the telescope has a vertical and horizontal wire the sun's image is made tangential to both. To get the lateral correction for semi-diameter we must multiply the Almanac value of the latter by the secant of the altitude. Both these corections are got rid of by observing the sun in opposite quadrants of the cross-wires. The best plan is to keep one edge of the sun tangential to one of the wires by means of a slow-motion screw till the other edge becomes also tangential. Thus, we might let the sun overlap the vertical wire a little and keep it tangential to the

horizontal wire by the vertical slow-motion screw till it just touches the former. If the wires of the theodolite are arranged as in Fig. 11 we take the observation as follows. Suppose the time to be forenoon and the apparent motion of the sun in the direction of the arrow. For the first observation get the sun tangential to the wires *a b* and *e f* in the uppper position. This is done by making it overlap *a b* a little, and using the vertical arc slow-motion screw to keep the lower edge tangential to the horizontal wire *e f* until the sun also touches *a b*, when the verniers are read off. The instrument is then reversed and the sun made tangential to the two wires in the lower position, this time using the horizontal plate slow-motion screw to keep the edge tangential to *a b*. The mean of the two altitudes is taken and also that of the two horizontal readings. The time must also be noted so as to correct the declination. The reading of the horizontal plate when the telescope is turned on the referring mark (A) is taken both before and after the sun observations.

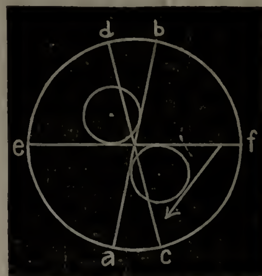


Fig. 11.

Ex. At Kingston in latitude 44° 13′ 40″ on the 3rd of March, 1882, at 2h. 30m. P.M., or 7h. 36m. Greenwich mean time, two altitudes were taken of the sun with a transit theodolite in reversed positions for the purpose of testing the accuracy of a north and south line, the horizontal arc being first clamped at zero, and the telescope directed northwards along the line.

READINGS ON SUN.

	Altitude.	Azimuth.
1st observation—	31° 8′	220° 0′
2nd "	30 16	220 16
Mean	30 42	220 8

To Correct the Altitude.		To Correct the Declination.	
	30° 42′ 0″	Declination at Mean Noon at Greenwich	6° 44′ 37″ S
Refraction—	1 36	Correction for $7\frac{1}{2}$ hrs. at—57″. 5 per hour	7 31 S
Parallax+	—8	True declination......	6 37 6 S
True altitude	30 40 32		90
		Sun's N. P. D.=	96 37 6

Formula used:

$$\text{Cos.}^2 \frac{Z}{2} = \cos. s. \cos. (s - \text{S P}) \sec. \lambda \sec. a$$

$a=$	30° 40′ 32″	log cos $s=$	8.8687314
$\lambda=$	44 13 40	log cos (s—SP)=	9.9921540
S P=	96 37 6	log sec a	=10.0654637
		log sec λ	=10.1447380
	2)171 31 18		2) 39.0710881
$s=$	85 45 39		19.5355440
	96 37 6		=10+log cos 69° 56′
s—S P=	10 51 27		2
			Z=139° 52
			360 0
		Sun's Azimuth=	220° 8′

The direction of the line was therefore true.

The formula—

$$\text{Cos}^2 \frac{\text{PZS}}{2} = \cos s \cos (s - \text{PS}) \sec \lambda \sec a$$

where PS is the sun's (or star's) polar distance, a its altitude, λ the latitude, and $s = \frac{\text{PS} + \lambda + a}{2}$, is thus derived.

We have, in the triangle PZS, if $s' = \frac{\text{PZ} + \text{PS} + \text{ZS}}{2}$

$$\text{Cos}^2 \frac{\text{PZS}}{2} = \frac{\sin s' \sin (s' - \text{PS})}{\sin \text{PZ} \sin \text{ZS}}$$

$$\text{Sin} (s' - \text{PS}) = \cos \{90 - (s' - \text{PS})\}$$

$$=\cos\frac{180-PZ-ZS+PS}{2}$$

$$=\cos\frac{90-PZ+90-ZS+PS}{2}$$

$$=\cos\frac{\lambda+\alpha+PS}{2}=\cos s$$

$$\cos(s-PS)=\cos\frac{90-PZ+90-ZS-PS}{2}$$

$$=\cos\left\{90-\frac{PZ+PS+ZS}{2}\right\}=\sin s'$$

Therefore $\cos^2\frac{PZS}{2}=\frac{\cos s\cos(s-PS)}{\cos\lambda\cos\alpha}$

Similarly it may be shown that

$$\sin^2\frac{ZPS}{2}=\frac{\cos s\sin(s-\alpha)}{\cos\lambda\sin PS}$$

TO FIND THE MERIDIAN BY EQUAL ALTITUDES OF A STAR.

Select a star which describes a good large arc in the sky, and having levelled the theodolite direct the telescope on it about two hours before it attains its greatest height. Clamp both arcs, and by means of the slow motion screws get the star exactly at the intersection of the wires. Having taken the reading of the horizontal arc, leave the vertical one clamped, loosen the upper horizontal plate, and look out for the star when it has nearly come down again to the same altitude. When it enters the field of view follow it with the telescope, using the horizontal slow-motion screw, but still keeping the vertical arc clamped, till it is exactly at the intersection of the wires. Now read the horizontal plate: the mean of the two readings will give the direction of the meridian. Set the plate at that reading and send out an assistant with a lantern. Get the latter exactly at the intersection of the wires, and drive in pickets at the lantern and theodolite station.

if great accuracy is not requ

This method is rather a tedious one, but it may be shortened by observing the star when nearer the meridian.

TO FIND THE MERIDIAN BY AN OBSERVATION OF THE POLE STAR AT ITS MERIDIAN TRANSIT.

Ascertain the watch error by any of the ordinary methods and calculate the exact instant at which the pole star will be on the meridian, either above or below the pole. If the theodolite telescope is directed on it at this instant we shall evidently have the meridian line, provided the instrument is in good adjustment. But it is better, in order to eliminate instrumental errors, to proceed as follows: at some definite time before the star will be on the meridian—say 2 minutes—direct the telescope on it and take the reading of the horizontal plate. Reverse the telescope and horizontal plate, and direct the former on the star at the same interval of time after the transit—in this instance 4 minutes after the first observation—and again read the plate. The mean of the two readings will give the true north.

It is almost unnecessary to remark that in observations of this kind both verniers should be read.

TO FIND THE MERIDIAN BY THE GREATEST ELONGATION OF A CIRCUMPOLAR STAR.

This is a very accurate method. Stars which, like the pole star, are very near the pole, owing to their slow motion appear to move vertically for some little time when at their greatest eastern or western elongation. We will suppose the pole star to be the one observed. About six hours before or after the transit (the time of which must be previously calculated) the theodolite is carefully levelled, its telescope directed on the star, and the horizontal plate read. The operation is then repeated in reversed positions of the instrument and the mean of the two readings taken. If we have previously taken the horizontal plate reading when the telescope was turned on some well-defined distant object as a referring work we can now obtain the azimuth of the latter as follows:

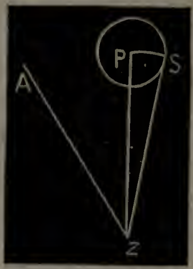


Fig. 12.

In Fig. 12 let the plane of the paper represent the plane of the horizon, and let Z be the observer's position, A the referring mark, P the pole, and S the star at its greatest eastern elongation. P Z S will be a spherical triangle, right-angled at S, and we shall have:

$$\text{Sin P } Z \text{ S} = \frac{\text{Sin P S}}{\text{Sin P } Z}$$

or, if δ is the star's declination and λ the latitude of the place:

$$\text{Sin P} Z \text{S} = \frac{\text{Cos } \delta}{\text{Cos } \lambda}$$

since P S is the complement of the declination, and P Z the complement of the latitude. The latitude need not be very accurately known.

Now, having from this equation previously found the angle P Z S, and having obtained A Z S from the plate readings, we get at once the angle A Z P, which is the required azimuth.

If we have to use a star some distance from the pole we must calculate the time of its greatest elongation by solving the equation.

$$\text{Cos } Z \text{ P S} = \text{cot } \delta \text{ tan } \lambda,$$

which gives us the star's hour angle, and hence the time of the observation.

The altitude is given by the equation.

$$\text{Sin. altitude} = \frac{\sin \lambda}{\sin \delta}$$

If it is inconvenient to observe the pole star at its greatest elongation we can use the following formula which is approximately true in the case of a star very near the pole.

$$\frac{\tan A^1}{\tan A} = \sin Z \text{ P S}$$

where A^1 is the star's azimuth, Z P S its hour angle at the time of observation, and A its azimuth at greatest elongation.

TO FIND THE MERIDIAN BY OBSERVATIONS OF HIGH AND LOW STARS.

This is a very useful method, as it is independent of the pole star, and can therefore be employed in the southern hemisphere where that star is not visible.

Choose two stars differing but little in right ascension, one of which culminates near the zenith and the other near the south horizon (or the north horizon if in the southern hemisphere.) Level the theodolite very carefully. The great circle swept out by the collimation line of the telescope will coincide with the meridian at the zenith, however far it may be from it at the horizon; and a star culminating near the zenith will cross the centre of the field of view at nearly the same time as if the telescope had been truly adjusted in the meridian, while this will not be the case with a star culminating near the horizon. Having calculated the true times at which the stars will cross the meridian observe the transit of the upper star, noting the watch time. This will give the watch error approximately, and we shall now know the approximate watch time at which the lower star will transit. By keeping the telescope turned on that star till that instant arrives we shall get it very nearly in the plane of the meridian; and by repeating the process with another pair of high and low stars we shall have the direction of the meridian with great exactness.

For this method we require a transit theodolite fitted with a diagonal eye piece. The nearer the upper stars are to the zenith the better.

The Canadian Government Manual of Survey recommends for azimuth the formula:

$$\tan.\ P\,Z\,S = \frac{\tan P\,S \sec \lambda \sin Z\,P\,S}{1 - \tan P\,S \tan \lambda \cos Z\,P\,S}$$

as applied to observations of the pole star; but it requires special tables in order to work it out.

The following is the proof of this formula:

We have the fundamental formulœ—

$$\begin{cases} \sin a \sin C = \sin c \sin A & (1) \\ \cos C = \dfrac{\cos c - \cos a \cos b}{\sin a \sin b} & (2) \\ \cos A = \dfrac{\cos. a - \cos b \cos c}{\sin b \sin c} & (3) \end{cases}$$

From (3), $\cos a \cos b = \cos^2 b \cos c + \sin b \sin c \cos b \cos A$

From (1 & 2), $\cot C = \dfrac{\cos c - \cos a \cos b}{\sin a \sin b \sin C}$

$$= \frac{\cos c - \cos^2 b \cos c - \sin b \cos b \sin c \cos A}{\sin b \sin c \sin A}$$

$$\therefore \tan C = \frac{\sin b \sin c \sin A}{\sin^2 b \cos c - \sin b \cos b \sin c \cos A}$$

$$= \frac{\operatorname{cosec} b \tan c \sin A}{1 - \cot b \tan c \cos A}$$

In the triangle P Z S let P S$=c$, Z S$=a$, and P Z$=b$
Z=C and P=A

Then—

$$\tan PZS = \frac{\operatorname{cosec} PZ \tan PS \sin ZPS}{1 - \cot PZ \tan PS \cos ZPS}$$

$$= \frac{\tan PS \sec \lambda \sin ZPS}{1 - \tan PS \tan \lambda \cos ZPS}$$

CHAPTER V.

SUN DIALS.

To a person acquainted with the rudiments of Astronomy a little consideration will show that if a straight line—such as the straight edge of a thin plate of metal—is placed parallel to the polar axis of the earth, its shadow thrown on any plane surface will, for a given hour angle of the sun, always lie in the same straight line whatever be the sun's declination. The shadow of any particular *point* in the line will move as the declination varies, but will always lie in the same straight line for any given hour angle. On this principal all sun-dials are constructed. The position of the shadow shows the sun's hour angle at the instant, and therefore indicates the *apparent* solar time; so that in order to obtain ordinary *mean* time we have to apply the equation of time.

Dials are generally either horizontal or vertical. In the former case the shadow of the stile, as it is called, is thrown on a horizontal plate; in the latter on a vertical wall.

The simplest forms of horizontal dials are those which would be employed at the equator and at the poles. At the equator the dial would evidently consist of a vertical plate having a horizontal edge, lying north and south. The shadow lines would be parallel to the stile, and their distances apart for equal intervals of time would rapidly

increase according to the sun's distance from the meridian, and would become indefinitely great when he was on the horizon.

At the poles the stile would be a fine vertical rod, from the base of which 24 straight lines, radiating at intervals of 15 degrees, would indicate the hours. The line on which the shadow was thrown at the time corresponding to Greenwich mean noon might be assumed as the zero or 24-hour line. At other places the stile must be set so that its angle of elevation above the horizontal plane is the same as the latitude of the place.

HORIZONTAL DIALS.

A horizontal dial generally consists of a triangular metal stile fixed on a horizontal plate on the top of a pillar. Fig. 13 is an elevation and Fig. 14 a plan. The angle of elevation of the stile is made equal to the latitude of the place, and if the variation of the compass is known, the latter may be used to get the dial with its stile in the plane of the meridian. The hour lines on the plate are marked out thus: let A B (Fig 14) be the base of the stile,

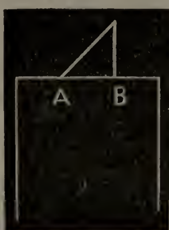


Fig. 13.

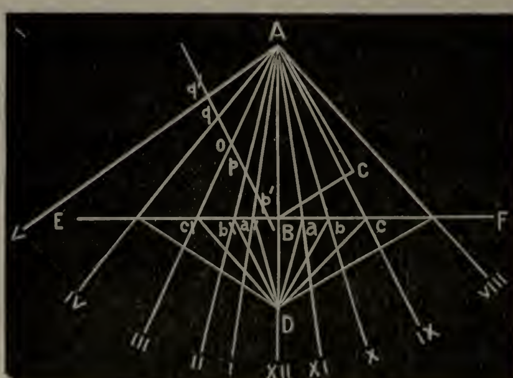


Fig. 14.

and A its south end. Draw A C so that B A C is equal to the latitude, and at any point C in A C draw C B perpendicular to A C, meeting A B in B; produce A B to D and make B D equal to B C. Through B draw a straight line E B F perpendicular to A B D. From D draw lines D a, D a^1, D b, D b^1, &c., meeting E B F in a, a^1, &c., and making the angles B D a, B D a^1, a D b, a^1 D b^1, &c., each equal to 15 degrees. From A draw straight lines through a, b, a^1, b^1, &c. These will be the hour lines: A c for 9 A. M., A b for 10 A. M., A a for 11 A. M., and so on. The proof of the correctness of this construction is easily seen by imagining the triangle A B C to be turned round A B till it is perpendicular to the plane of the paper or dial plate, and the triangle c D c^1 to be turned up on c c^1 till it abuts on B C, when D will coincide with C, and A C will be parallel to the polar axis and perpendicular to the plane of D c c^1.

When the divisions on the line E B F run off the plate we continue them thus: In A c^1 (the 3 P. M. line) take any point o, and through it draw a line parallel to A c, (the 9 A. M. line) meeting A b^1, A a^1 &c., in p, p^1, &c., and make o q equal to o p, o q^1 to o p^1, &c., and through q, q^1, &c., draw straight lines A q, A q^1 &c., which will be the evening hour lines.

A similar construction serves for the morning hour lines on the other side.

VERTICAL DIALS.

These have the advantage that they may be made of a very large size and placed in conspicuous positions. There are various ways of constructing them. As simple a plan as any is to fix a flat disk, having a round hole in it, in front of a wall with a southerly aspect. (Fig. 15.)

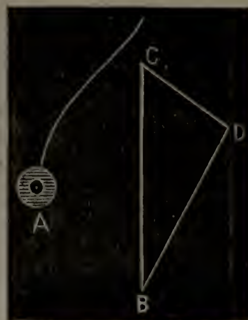


Fig. 15.

The disk should be roughly perpendicular to the sun's rays at noon about the equinoxes. ~~The~~ The bright spot in the middle of the shadow of the disk on the wall indicates the hour. The hour lines are found thus: At the time the sun is on the meridian mark the position of the bright spot on the wall. Let A be the hole in the disk and B the spot. Measure A B. Through B draw B C vertical, and draw a line B D so that B D is equal to A B, and the angle C B D to the sun's polar distance minus the co-latitude. Make the angle B D C equal to the supplement of the sun's polar distance. It follows from this construction that if the triangle B C D were turned round B C till it touched A the points D and A would coincide, and C D (and therefore the imaginary line C A) would be parallel to the polar axis. Now take a watch, set to noon at the time of the sun's transit, and mark the positions of the spot on the wall at the successive hours. Straight lines joining these points with C will be the hour lines.

Of course a large triangular stile C A B might be substituted for the disk; or we might use a rod C A fixed in the plane of the meridian, and having the angle A C B (which it makes with the vertical) equal~~s~~ to the co-latitude.

CHAPTER VI.

REMARKS ON PORTABLE ASTRONOMICAL INSTRUMENTS.

THE REFRACTING TELESCOPE.

The refracting telescope in its simplest form is a tube with an object glass at one end and an eye piece at the other. The former produces an inverted image of the object at its focus in the same way as the lens of a photographic camera, and the eye piece simply serves to magnify this image. The eye piece usually has two lenses; but if the telescope is wanted to show the object in its natural position a combination of four lenses is employed, by means of which the inverted image is again inverted. This has, however, the disadvantage of cutting off too much light, and is only used in small telescopes and for land objects.

The ordinary eye piece is of two kinds; first, the "negative," in which the image is formed between the two lenses of the eye piece. This is the kind generally used in telescopes designed for the mere examination of objects without making measurements. Secondly, the "positive," in which the common focus of the two lenses is outside the eye piece. This is the kind used in all telescopes intended for measuring angles by means of spider lines, &c. When the instrument is properly focused the image and system of wires are in the same

plane at the common focus of the object glass and eye piece. The position of the focus of the former depends on the distance of the object—that of the latter on the eye of the observer. The one is the same for every individual. The other has to be adjusted to suit the observer—short-sighted people having to push the eye piece in, while those who have long sight require a longer focus.

The larger the object glass is the more rays from the object are collected on the image, and the brighter it is. The greater the magnifying power of the eye piece the more apparent are any defects of definition in the image.

The *magnifying power* of the telescope is measured by the fraction $\frac{\text{focal length of object glass.}}{\text{focal length of eye piece.}}$ Thus, if this fraction were 4, the linear dimensions of the object seen through the telescope would be four times what they would be when viewed with the naked eye. Therefore, for a given eye piece, the longer the telescope is the smaller will be the *field of view*, or portion of the earth or sky visible. The angular diameter of the field is, in fact, the angle subtended by the diameter of the eye piece at the centre of the object glass. that the longer the

In large telescopes the field of view is so small that it is necessary to use a "finder," which is simply a small telescope attached to it so that the axes of the two shall be parallel.

A *diagonal eye piece* is one in which there is a mirror or prism between its two lenses by which the rays of light are turned at right angles and emerge from the side instead of the end of the eye piece. It is used for observing objects when the altitude is so great that it would be uncomfortable or impossible to look up through the telescope tube.

Lenses have to be corrected for *chromatic aberration* and *spherical aberration*. Take the case of an object glass con-

sisting of a simple glass lens. Rays of light of different colour having different degrees of refrangibility would have different foci. The result would be indistinctness of image, and, if the object were white, a sort of rainbow appearance of coloured light. This chromatic aberration is got rid of by using a compound lens consisting of two lenses of glass of different refrangibility, one behind the other. Such a lens is called "achromatic."

By "spherical aberration" is meant the dispersion of rays caused by the central portion of a lens with spherical surfaces having a different focus from its outer or marginal part. This is corrected by the form given to the aberration combination.

The way to ascertain if the object glass has been properly corrected for colour is to turn it on some bright object, such as the moon or Jupiter, and put the eye piece a little out of focus. This ought to produce either a purple or pale green ring round the image, according as the lens is pushed in or drawn out.

Spherical aberration is detected by covering the central portion of the lens with a circular disk of paper and focusing it on an object, afterwards removing the disk and covering the outer part with a ring-shaped piece, when the focus ought to remain the same.

If one part of the object glass has a different refractive power from another part a bright star will appear with an irradiation, or *wing*, at one side.

THE MICROMETER.

The micrometer is a contrivance for measuring small angular distances. It is, like the cross wires of an ordinary theodolite, placed in the common focus of the object glass and eye piece of a telescope, or of the "reading microscope," which will be described presently. The principle of the micrometer is simply this: Suppose that

a point—such as the intersection of the cross wires—can be moved across the field of view by means of a screw. Let a be the angular diameter of the field, and let n be the number of complete turns of the screw required to move the wires through this space; it is evident that one turn of the screw will move them through an angle $\frac{a}{n}$ The head of the screw should of course have an index to mark the commencement and end of each turn. If the head is made large enough to enable its rim to be divided into m equal parts we shall have the means of measuring an angle $\frac{a}{m\,n}$ by turning the head through one division. Thus, by making the thread of the screw sufficiently fine, and its head large enough, we have the means of measuring small angles to an éxtreme degree of accuracy, provided we know the angular value of one turn of the screw. This may be ascertained by finding how many turns it takes to move the wire across the image of an object of known dimensions at a known distance. A levelling rod will answer the purpose. The length of rod moved over, divided by the distance, gives, of course, the chord of the angle subtended. There is usually a scale in the field of view, the divisions of which correspond to the turns of the screw.

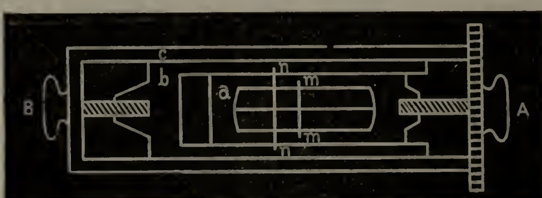


Fig. 16.

There are several different forms of micrometer. The accompanying figure (16) represents the one known as the *filar* (or thread) micrometer. Two parallel wires $m\ m$,

n n, are fixed to two frames, *a*, *b*, sliding one within the other, which are moved by screws as shown. The frame *b* slides within an outer fixed frame *c*. The frame *a* carries the wire *m m*, and is moved by the screw A, the head of which has a graduated rim. *b* carries the wire *n n* and is moved by the screw B. In the oval opening of *a* is a horizontal wire at right angles to the others. Suppose we have to measure the angular distance between two stars. The instrument must be turned till both are on the horizontal wire. If now the screw A is turned till *m m* cuts one star, and B till *n n* cuts the other, the distance between the two is measured by the number of turns of A and fractional divisions of a turn it takes to bring *m m* up to *n n*. This is not the exact method of procedure followed, but it serves to illustrate the principle.

THE READING MICROSCOPE.

The *reading microscope* is really a small telescope with a micrometer in its focus, and is used instead of a vernier for reading the fractional parts of the divisions of the graduated circles of large instruments. The microscope is fixed, the circle being attached to the telescope of the instrument and moving with it. The micrometer has only one screw and moveable frame, the latter carrying a pair of cross-wires in the common focus of the object glass and eye piece of the microscope. These cross-wires are used in exactly the same way as those of a theodolite telescope, only that the object viewed is the graduated arc, on which the microscope must, of course, be focused. To make the matter clear we will take the case of the measurement of a horizontal angle by a large theodolite the arc of which is graduated to 10 minutes. Suppose that one turn of the micrometer screw is equivalent to one minute, and that its head is divided into 60 parts. We have thus the means of measuring the angle to single seconds. The circle with its attached telescope is revolved, and the cross-wires of the latter made to coincide

with one of the objects. On viewing the arc through the microscope (which it must be remembered is a fixture) the wire intersection of the latter must be made to coincide with one of the divisions of the arc by means of the micrometer screw, and the reading of the index of the latter noted. Suppose the arc reading to be $10° 20'$, and that of the screw head $15''$. Now move the circle and bring the telescope to bear on the other object. The cross-wires of the microscope will probably fall somewhere between two divisions of the arc, say between $50°$ $30'$ and $50° 40'$. Turn the screw till the cross-wire is on the $50° 30'$ division, and suppose that it takes between three and four turns, and that the index marks $25''$. The micrometer wire will have been moved $3' 10''$, and the true reading of the second object will be $50° 33 10''$. The angle measured is therefore $40° 13' 10''$.

Two or more reading microscopes are placed at equal distances round the circle, and the readings of all taken. Errors due to eccentricity are thus got rid of, and those due to faulty graduation and observation much diminished.

THE SPIRIT LEVEL.

The spirit level is used, not only to bring certain lines of an instrument as nearly as possible into a horizontal position, but also to measure the deviation of these lines from the horizontal. For this purpose the glass tube is graduated, usually from its middle towards both ends, and the reading of the ends of the air bubble noted. The length of the bubble depends upon the temperature, and the latter should therefore be also noted.

To obtain the *value of one division of the level*—that is, of the vertical angle through which the level must be moved in order that the ends of the bubble may be displaced one division—a simple plan is to rest the level upon some support (such as the horizontal plate of a theodolite) that can be moved vertically and which is con-

nected with a telescope. The plate is levelled, the telescope directed on a vertical measuring rod set up at a known distance, and readings taken of the ends of the bubble and of the intersection of the horizontal wire with the rod. The whole arrangement is then moved vertically by means of the foot screws till the ends of the bubble have moved a certain number of divisions—say 10. The reading of the telescope wire on the rod is now noted. The difference of the rod readings, divided by its distance, gives the chord of the vertical angle moved by the level, and this divided by 10 gives the value of one division.

In the case of the level of the transit telescope at the Royal Military College it was found that at a distance of 383 feet a vertical movement that displaced the bubble 20 divisions altered the reading on a levelling staff 0.24 feet. The resulting decimal gave 0.0000313 as the tangent of the subtended angle for one division, which made the value of the latter 6″.45.

In the case of a striding level, if its feet, or the surface on which they rest, are truly horizontal, while one leg is longer than the other, the bubble reading of the end which is highest will be greatest, and if the level is turned end for end the bubble readings will change places. On the other hand, if the legs are of equal length, but the surface on which they rest is not level, the bubble readings will remain the same on reversal. That is, if we call one end of the level A and the other B, the level error will make the reading of A the largest in both positions. If a truly adjusted level rests on a slope, and A reading is the greatest in one position, B will be greatest by the same amount on reversal.

In practice we generally find both errors combined, especially in the case of the more delicate levels which easily get out of adjustment. The amount of slope of the surface tested by the level is obtained thus: Take the

case of the pivots of a transit telescope. Placing the level upon them take the readings of the bubble ends, and call the reading next the west pivot W and the other E. Then reversing the level take the readings over again and call them W^1 and E^1. The number of divisions by which the bubble is displaced by the difference of level of the pivots is given by the formula:

$$\frac{W+W^1-(E+E^1)}{4}$$

To find the *actual slope* of the pivots we must multiply this quantity by the value of a division of the level.

The *level error* is obtained separately by simply changing the signs of W^1 and E^1 in the above formula, when we have:

$$\text{Level error} = \frac{W-W^1-E+E^1}{4} = \frac{W-E-(W^1-E^1)}{4}$$

Of course if $W+W^1=E+E^1$ there is no slope, and, in practice, when the level is out of adjustment, we may get the points of support horizontal by raising one of them till this is the case: For instance; if W were 20 and E 10, W^1 would have to be 10 and E^1 20.

If the level is in adjustment we must have

$$W-E=W^1-E^1$$

In this case we have only to take W and E and the slope is obtained from the formula

$$\frac{W-E}{2} \times \text{value of one division}$$

Example—Take $W=25$, $E=10$, $W^1=15$, $E^1=20$. Value of one division $=6''$.

$$\text{Here we have } \frac{25+15-10-20}{4} = \frac{10}{4} = 2\tfrac{1}{2}$$

Multiplying this by $6''$ we have $15''$ as the slope, the west pivot being the highest.

$$\text{The level error is } \frac{25-15-10+20}{4} \times 6'' = 30''$$

If great accuracy is required the level should be read a number of times in each position, lifting it up after each reading and taking the same number of readings in the two positions. The difference of the sums of all the readings at the two ends divided by the total number of the readings will give the slope.

The value of a division of the level should be ascertained for different temperatures.

To correct the error of the level get the surface on which it rests truly horizontal as explained above. Then, by means of the adjusting screw, move the bubble till both its ends read the same.

THE CHRONOMETER.

The chronometer is simply a very perfect watch without a regulator, and with the balance so constructed that changes of temperature have the least possible effect upon the time of its oscillation. Chronometers may be constructed to keep either mean or sidereal time. Those used on board British ships are intended to show Greenwich mean time. The great point in a chronometer is that it should keep a regular rate; that is, that it should only gain or lose a certain amount in a given time. If this can be depended on we can always ascertain the true time at any instant by applying the rate for the number of days and hours that have elapsed since the actual error of the chronometer was last determined, whether by comparison with other chronometers or by an astronomical observation. The more regular the rate kept the more perfect is the chronometer. It is also more convenient to have a small than a large rate to allow for.

Chronometers are generally made to run either two or eight days. The former are wound daily, the latter every seventh day. It is important that they should be wound up at the regular intervals, as, if let to go too long, an unused part of the spring comes into play, and irregu-

larity of rate may result. If a chronometer has run down it requires a quick rotatory movement to start it after it has been wound.

Transporting—On board ship chronometers are allowed to swing freely in their gimbals so that they may keep a horizontal position ; but on land they should be fastened with a clamp. Pocket chronometers should always be kept in the same position, and if carried in the pocket in the day should be hung up at night.

Chronometers have usually a different rate when travelling from what they keep when stationary. The travelling rate may be found by comparing observations for time taken at the same place before and after a journey, or from observations at two places of which the difference of longitude is known.

For mean time observations an ordinary watch may be used by comparing it with the chronometer, provided the rate of the watch is known.

Chronometers are generally made to beat half seconds.

THE ELECTRO CHRONOGRAPH.

Under this head may be included all contrivances for registering small intervals of time by visible marks produced by an electro magnet, and thus recording to a precise fraction of a second the actual instant of an occurrance. By this means an observer at a station A can record at a distant station B the exact instant at which a given star passes his meridian, and thus the difference of longitude of the two stations may be ascertained.

REFLECTING INSTRUMENTS.

THE SEXTANT.

A person accustomed to work with the pocket sextant will have little difficulty in using the larger kind ; and the latter, with its adjustments, is so fully described in most

works on surveying that little need be said about it here. On land it is chiefly used for taking altitudes with an artificial horizon. The latter usually consists of a trough of mercury with a glass roof to protect it from the wind, or with a plate of glass floating on the mercury. The trough should be a sufficiently large one, otherwise the observer will be continually losing time and patience by failing to catch both images together. To eliminate errors caused by want of parallelism in the glass of the roof, when one half of a set of observations has been taken the roof should be reversed end for end. For taking the sun's double altitude the dark glass of the eye piece may be used. The two images should be made to overlap each other a little, the vernier clamped, and the instant noted when the circles just touch. As this requires that the images should be receding from each other, the altitude of the lower limb must be taken in the forenoon and of the upper limb in the afternoon. For a lunar distance of the sun direct the telescope on the moon and use one or more of the hinged dark glasses for the sun. The index error should be obtained, and applied as a constant correction.

A common fault of the sextant is that the optical power of the telescope is too small. There is little use in being able to read the graduation to 10 seconds if the eye cannot be sure of the contact of the images within 30″.

THE SIMPLE REFLECTING CIRCLE.

This is simply a sextant with its arc graduated for the whole circumference of a circle, and with the index arm produced to meet it at opposite points and carrying a vernier at each end. The mean of the two verniers can be taken at each observation and any error due to eccentricity thus got rid of. This arrangement also tends to diminish the errors of graduation and observation.

Some reflecting circles have three verniers at intervals of 120°.

THE REPEATING REFLECTING CIRCLE.

In the repeating reflecting circle the horizon glass (*m* Fig. 17), instead of being immovable, is attached to an arm which revolves about the centre of the instrumeut and which also carries the telescope (*t*) and a vernier (*v*). The index glass (~~*m*~~ M) is carried on another revolving arm, which also has a vernier v^1. The arc is graduated from 0° to 720° in the direction of the hands of a clock. To use the instrument the index arm is clamped and its reading taken. The telescope is then directed on the right hand object (*b*), the circle revolved till the images coincide, and the telescope arm clamped. The index arm is then unclamped, the telescope directed on the left hand object (*a*), and the index moved forward till the images again coincide, when its vernier is read. The difference between the two readings of the index vernier is twice the angle between the objects. This repeating process may be carried on for any even number of times. The first and last readings only are taken, and their difference, divided by the number of ~~repetitions~~ contacts, gives the angle. If the angle is changing, as in the case of an altitude, the result will be the mean of the angles observed, and the time of each observation having been noted the mean of the times is taken.

Fig. 17.

This instrument will not measure a greater angle than the sextant. Its advantages over the latter are that there is no index error, and errors of reading, graduation, and eccentricity are all nearly eliminated by taking a sufficient number of cross-observations.

Fig 17. refers to the dip measurer.

The *dip-measurer* is a repeating circle which has a third mirror (*n*) mounted on the telescope arm at an angle of about 45° to the horizon glass, but only one-half the height of its silvered portion. By means of this modification angles of even more than 180° can be measured, and the instrument can therefore be used for taking double altitudes of objects near the zenith.

THE PRISMATIC REFLECTING CIRCLE.

In the prismatic reflecting circle the telescope (*t*) is fixed, and instead of the horizon mirror there is a small fixed prismatic reflector (*p*) which half covers the object glass. The index mirror (*m*) is carried on an arm which revolves round the centre of the circle and has a vernier (v v^1) at both ends. This instrument will measure angles of any dimension, and has also the following advantages: (1) Eccentricity is completely eliminated by using both verniers. (2) The reflected images are brighter than in the case of other reflecting instruments. (3) The prismatic reflector causes less error than the ordinary one.

Fig. 18.

At angles about 180° the observer's head gets in the way, when a prismatic eye piece is used.

The *prismatic sextant* differs from the circle in having only one vernier, and in the arc not extending the whole circumference. It measures the same angles as the circle, but does not eliminate the eccentricity.

(For a full description of the above reflecting instruments, with their adjustments and the method of using them, *vide* Chauvenet's Astronomy.)

CHAPTER VII.

THE PORTABLE TRANSIT INSTRUMENT.

The transit instrument is a telescope with two trunnions resting on Y-shaped supports so that its line of collimation may move in a vertical plane, and is used for the purpose of taking the times of transit of heavenly bodies across that plane—generally either the meridian or the prime vertical. In the former case it enables us to find the true local time, either mean or sidereal, and also serves to determine the longitude by means of transits of moon-culminating stars. In the latter case it gives us a very accurate method of ascertaining the latitude by transits of stars across the prime vertical.

In the focus of the telescope are one or two horizontal wires, and an odd number of equi-distant vertical ones—generally five—of which the central should be in the optical axis of the instrument, and at right angles to the axis of the trunnions or pivots; and if, in addition, this axis is truly horizontal, the line of collimation will move in a vertical plane. The telescope is provided with a vertical graduated circle, with a level attached, which serves as a finder to set it at any required angle of elevation. It has also a diagonal eye piece for transits of objects of considerable elevation, and a very delicate striding level for getting the pivots perfectly horizontal. At night the light

of a lantern is thrown into the interior to illuminate the wires by means of an opening with a lens in one of the pivots and a small internal reflector. In taking transits it is usual to note the time of the object's passing each wire, and to use the mean of all the wires instead of the transit across the central one. In the field the best plan is for an assistant to hold the watch or chronometer, the observer calling out "stop" as the object passes each wire. In the case of the sun, moon, and planets the instant noted is when the edge of the object touches the wire, either in coming up to it or leaving it, the time required for its semi-diameter to pass the meridian being afterwards added or subtracted. unless both limbs are observed in which case the mean is taken

The first adjustment to be attended to is that of collimation. This may be effected by getting the central wire on some well-defined distant object, or on a circumpolar star at its greatest elongation. The telescope is then reversed in its supports, end for end, when, if the wire still bisects the object, the collimation is all right. If it falls to one side of it it must be moved towards it half the interval by the collimation screws. The instrument is then moved laterally by means of small screws connected with one of the Y supports till the wire bisects the object, when the telescope is again reversed and the process repeated till the collimation is perfect.

The horizontality of the axis of the pivots is obtained by the striding level and foot screws. As the level has generally an error of its own which is itself liable to change (owing to alterations in temperature, accidental flexure, &c.) it will be found convenient to level the pivots by getting them in such a position that the level will have equal but opposite readings in reversed positions. Thus, if in one position the east end of the bubble reads 10, while the west end reads 12, then, on reversal, the east end should read 12 and the west 10.

If one of the pivots has a larger diameter than the other it is evident that when their upper surface is level their axis will not be so. This will entail a constant error which will be investigated presently.

The verticality of the central wire must be tested by levelling the pivots and noticing whether the wire remains upon the same point throughout its whole length when the telescope is slowly moved in altitude.

If the collimation is out of adjustment, but the levelling correct, the line of collimation will sweep out a cone. If the collimation is correct but the levelling inaccurate, it will describe a great circle, but not a vertical one. If both are right it will move in a vertical plane. We have now to make this plane coincide with some given one—say that of the meridian. The north and south line may have been already approximately obtained by means of a theodolite, and we can now find it exactly by one of the following methods.

(1) By transits of two stars differing little in right ascension, one as near the pole, the other as far from it as possible. Let α be the right ascension, δ the declination and t the observed clock time of transit of the star near the pole; α^1, δ^1, and t^1 the same quantities for the other star, d the azimuth of the instrument—in other words, the error or deviation to be determined—and φ the latitude. Then d is found from the formula,

$$d = \left\{ \overset{(t'-t)-(\alpha'-\alpha)}{\cancel{(\alpha^1-\alpha)-(t^1-t)}} \right\} \frac{\cos\delta\cos\delta^1}{\cos\varphi\sin(\delta-\delta^1)}$$

The rate of the clock must be known, but not its error; the interval t^1-t must be corrected for error of rate; and, if a mean time watch is used, converted into sidereal time. d being in horary units must be multiplied by 15 to obtain the error in arc.

If the declination of the southern star is south it will,

of course, be negative in the northern hemisphere, and *vice versa.* $(t'-t)-(a'-a)$ is negative. t' &c refers to stars below the pole.

If ~~the interval between the transits is too small~~ the deviation will be west of north, and *vice versa.* In the southern hemisphere the contrary would be the case.

(2) By observing the transits of two circumpolar stars which culminate within a short time of each other, one above, the other below the pole. δ Ursœ Minoris and 51 Cephei are a good pair for this. These stars being very slow moving it is evident that a slight deviation of the instrument will cause one to come to the central wire too soon, and the other too late. The formula is

$$d = \left\{ \begin{array}{c} (t'-t)-(a'-a \pm 12h) \\ \cancel{a^1 - a - 12h - (t^1 - t)} \end{array} \right\} \frac{\cos\delta\cos\delta^1}{\cos\varphi\sin(\delta+\delta^1)}$$

t' &c refers to the lower star.

The value of the fractional part of this formula at Kingston (Lat. 44° 14′) is 0.0372 for the above pair of stars.

If ~~the interval between the two transits is too great,~~ $(t'-t)-(a'-a)$ is negative, the deviation is west of north and *vice versa.* It is always safest to draw a figure in these cases.

The deviation having been calculated the position of the meridian mark can be corrected if its distance from the instrument is known.

To prove the formula:

$$d = \left\{ a^1 - a - (t^1 - t) \right\} \frac{\cos\delta\cos\delta^1}{\cos\varphi\sin(\delta-\delta^1)}$$

Let P be the pole, Z the zenith, A^1 Z B^1 the plane in which the telescope moves, and A Z B the true meridian. We have to find the angle A Z A^1.

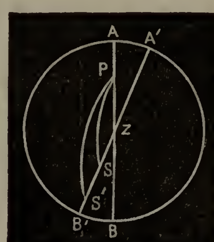


Fig. 19.

Let S and S^1 be the two stars at transit, e the unknown clock error, t the clock time when the star S was on

the meridian. The true time of the star being at S will be $t+e$.

Let α be the R. A. of the star. Then α was the time when the star was on P B; $\therefore$ Z P S$=t+e-\alpha$.

Let φ be the observer's latitude, δ the star's declination, d the deviation of the instrument.

From the triangle P Z S we have:

$$\text{Sin S Z sin S Z B} = \text{Sin P S sin S P Z} \qquad (1)$$

$$\text{And } Z\,\text{S} = \text{P S} - \text{P } Z, \text{ very nearly}, = \varphi - \delta$$

$$\therefore \sin(\varphi-\delta)\sin d = \sin(t+e-\alpha)\cos\delta$$

$$\text{or } (\sin\varphi\cos\delta - \cos\varphi\sin\delta)\,d = (t+e-\alpha)\cos\delta$$

$$\text{or } (\sin\varphi - \cos\varphi\tan\delta)\,d = t+e-\alpha \qquad (2)$$

If α^1, δ^1 be corresponding quantities for star S^1 we have

$$(\sin\varphi - \cos\varphi\tan\delta^1)\,d = t^1+e-\alpha^1 \qquad (3)$$

In equation (3) t^1 includes the correction for the clock rate between the observations.

Subtracting (2) from (3) we have

$$d\cos\varphi(\tan\delta - \tan\delta') = t^1 - t - (\alpha' - \alpha) \qquad (4)$$

$$\text{Now } \tan\delta - \tan\delta' = \frac{\sin(\delta-\delta')}{\cos\delta\cos\delta'}$$

$$\therefore d = \left\{t^1 - t - (\alpha' - \alpha)\right\}\frac{\cos\delta.\ \cos\delta'}{\cos\varphi.\ \sin(\delta-\delta'}$$

It is evident from equation (4) that for a given value of d the quantity $t^1-t-(\alpha^1-\alpha)$ is larger as $\tan\delta-\tan\delta'$ is larger. In other words, one of the stars should be as near the pole and the other as far south as possible.

Equation (1) may be put in the form

$$\text{Sin } Z \text{ P S} = \frac{d\sin Z\,\text{S}}{\cos\delta}$$

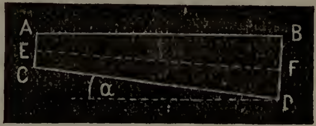
As the angle Z P S is the error, in time, of transit, caused by the azimuthal deviation d, this equation gives us the means of correcting a transit where it has not been convenient to correct the meridian mark.

TO FIND THE ERROR DUE TO INEQUALITY OF PIVOTS IN THE TRANSIT TELESCOPE.

In Fig. 20 let A C, B D, be the diameters of two unequal pivots, E F their axis. If the side A B on which the feet of the level rest is horizontal, the lower side

C D will be inclined at a certain angle, say a, and E F will be inclined at an angle $\frac{a}{2}$

Fig. 20.



If, now, the instrument be reversed in the Y's, A B will evidently be inclined at an angle $2a$, which will be given by the level readings in the usual way. Therefore, the inclination of the axis in the first position will be one quarter that given by the difference of the inclinations given by the level readings in the two positions. This quantity will be a constant correction to be applied to the inclination of the pivots as given by the level. For instance, in the case of the instrument at the Royal Military College the finder pivot was fonnd to be the thickest, moving the bubble 4 divisions on reversal. Therefore, when the upper surface of the pivots was level their lower surface was inclined for two divisions, and their axis for one. The level error has, therefore, to be corrected for the value of one division; *e. g.*, with west pivot highest by three divisions and finder east, the total error will be four. It reduces the error when the finder pivot is highest, and *vice versa.*

TO APPLY THE LEVEL CORRECTION TO AN OBSERVATION WITH THE TRANSIT TELESCOPE.

The level correction having been found in the usual way, let P be the pole, Z the zenith, S the star when on the central wire, R the south point of the horizon.

Fig. 21.

P R S is a spherical triangle in which P and R are very small angles. R is the level correction.

$$\text{Now}\ \frac{\text{Sin P S}}{\text{Sin R}} = \frac{\text{Sin S R}}{\text{Sin P}}$$

or

$$\frac{\text{cos. declination}}{\text{R}} = \frac{\text{sin. altitude}}{\text{P}}$$

$$\text{and P} = \frac{\text{R sin. altitude}}{\text{cos. declination}}$$

The correction for the transit in time will be $\frac{P}{15}$

Example—At Kingston, Canada, the transit of Arcturus was observed, the level readings being :

1st position	East 45—	West 35
2nd "	East 25—	West 55

To find the correction in time.

Here we have

Latitude	44°	14′	N
Star's declination	19	47	N
Star's altitude at transit	65	33	

$$\text{Level correction} = \frac{35+55-45-25}{4} = \frac{20}{4} = 5 \text{ divisions,}$$

west end being highest, and the pivot correction altered this to 6 divisions. The value of one division of the level was 6″.45, therefore the angle R was 38″.7 east of the meridian, and the transit took place too soon.

$P = \frac{91}{94} \times 38''.7 = 37''.2$, and the correction was 2.48s. to be added to the observed time of transit.

When the instrument is in perfect adjustment the error of the watch or chronometer can be at once obtained by means of meridian transits, as described at page 34.

FINDING THE LATITUDE BY TRANSITS OF STARS ACROSS THE PRIME VERTICAL.

If S is a star on the prime vertical, P the pole, *Z* the zenith, and W *Z* E the prime vertical, S P *Z* is a right-angled triangle ; and if we know the angle S P *Z* and the side P S we can find the side P *Z* by the equation

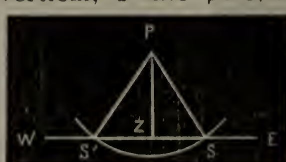


Fig. 22.

$$\text{Cos } S\,P\,Z = \tan P\,Z \cot P\,S$$

or, if a is the star's right ascension, t the sidereal time of its crossing the prime vertical, δ its declination, and φ the latitude of the place.

$$\text{Cos }(a-t)=\cot\varphi\tan\delta$$

If the star is in the position S′ (or west of the meridian) the equation becomes

$$\text{Cos }(t-a)=\cot\varphi\tan\delta$$

This method gives the latitude very accurately, provided the true sidereal time of the transit is known. The observation may be made either with a portable transit instrument or with a transit theodolite. In this, as in other delicate methods, the latitude is supposed to be approximately known beforehand. We can then from the above equation calculate the approximate time of the transit. The star's altitude is found from the equation

$$\text{Sin. altitude }=\frac{\sin\delta}{\sin\varphi}$$

$$(\text{Since }\cos PS=\cos PZ\cos ZS.)$$

It is evident that no stars cross the prime vertical except those whose declinations lie between zero and the latitude of the place. Those which culminate near the zenith are to be preferred, because a small error in the observed time of transit will affect the result less.

If the transit theodolite is used the true meridian line may be obtained by any of the ordinary methods, and the telescope clamped at right angles to it by means of the horizontal arc. If we are working with the portable transit telescope the method adopted is to calculate (from the approximate latitude) the time at which a star which culminates several degrees south of the zenith will cross the prime vertical, and direct the telescope on it at that instant. It will now be nearly in the required position. The error may be determined as follows: Take the sidereal time of transit of a star over both the

east and the west verticals. The mean of the two will be the time of the transit over the meridian of the instrument, and should be equal to the right ascension of the star. If the two results are not equal their difference shows the angle which the plane of the instrument makes with the true prime vertical.

In working these observations we may use either a sidereal or a mean time chronometer, in the latter case making the usual reductions, and always allowing for the rate. If two transit telescopes are available, one of them may be set up in the plane of the meridian for the purpose of ascertaining the exact chronometer or watch error by star transits. A large transit theodolite serves instead of two transit instruments, and in this case an ordinary good mean time watch will suffice, the mean time of the observations being reduced to sidereal time. If both the east and the west transits are observed the difference of time in sidereal units is double the hour angle P, and the latter may therefore be obtained without any reference to the actual watch error, provided the rate is known. It should also be noted that if we reverse the telescope on its supports any error of collimation or inequality of pivots will produce exactly contrary effects on the determination of the latitude. Two stars may be observed with the telescope in reversed positions on the same day, or the same star on two successive days, and the mean of the two resulting latitudes taken.

It will be found advisable to calculate beforehand the altitudes and times of transit (either mean or sidereal, as the case may be) of a number of suitable stars.

If the plane of the telescope is not in the prime vertical the calculated latitude will be too great. Suppose the deviation to be to the east of ~~north~~ and that the tele-

cope describes a vertical circle passing through ~~the~~ $E^1 Z W^1$. Then P Z^1, which bisects S S^1, will be the calculated co-latitude. The correction for the deviation may be computed thus. The star's R. A., minus the mean of the

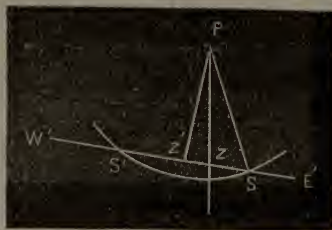


Fig. 23.

times of transit corrected for clock error, will be the angle Z P Z^1. Now, from the right-angled triangle Z P Z^1, we have:

$$\tan PZ \cos ZPZ' = \tan PZ' = \tan PS \cos SPZ'$$

$$\text{or} \qquad \tan \varphi = \frac{\tan \delta \cos ZPZ'}{\cos SPZ^1}$$

The angle ZPZ^1 is the same for all stars, and it is better to obtain its value from a star which culminates several degrees south of the zenith, since the same error in the observations will have less effect upon the calculated azimuth.

If the intervals between the vertical wires are not all exactly the same a correction has to be applied. For details on this point *vide* Chauvenet.

In the field the instrument is generally mounted on a wooden portable stand with three legs. In order to obviate the effects of vibration produced by the observer's movements the ends of the legs may be made to rest in notches in flat blocks of wood placed at the bottom of holes dug in the ground about eighteen inches deep and two feet in diameter. This was the plan adopted on the North American Boundary Survey between the Lake of the Woods and the Rocky Mountains.

The meridian mark should, if possible, be at least half a mile distant. A black or white vertical stripe painted on a stone serves for the day time. At night a bull's eye

lantern may be used, the glass being covered by a piece of tin with a vertical slit cut in it. Or, as the lantern is liable to be blown out by the wind, it may be enclosed in a wooden box with a vertical slit.

The larger transit theodolites may be used as transit instruments, and have the advantage over them that when the meridian line has been ascertained the prime vertical can be at once set off.

THE PERSONAL EQUATION.

It often happens that two persons, equally well trained in taking observations, will differ by a considerable and nearly constant quantity in estimating the precise instant of an event, such as the transit of a star across a wire. This difference is called their *personal equation*, and an allowance should always be made for it when observations made by two individuals have to be combined. In the case of the transit instrument this equation may be determined as follows: Let one observer note the passage of a star over the first three wires and the other observer note the transits over the remaining wires. If the two observers' estimation of the instant of transit differ, it is evident that (provided the wires are equidistant) the difference will appear on comparing the intervals of time. For instance, if A notes the transits across the first three wires at 10s., 20s., and 30s., and B notes the remaining two at 39s.5 and 49s.5, it is plain that A would consider the star to be on any wire half a second later than B would, and their personal equation is therefore 0s.5. By repeating the same process on other stars, and taking the mean of the result, a more accurate estimate is obtained. The personal equation has been found liable to vary with the state of health of the individual.

The difference in the estimated instant of a transit is only a particular case of the personal equation.

CHAPTER VIII.

THE ZENITH TELESCOPE.

The zenith telescope is a contrivance for the exact determination of the latitude by measuring with the greatest minuteness the differences of the meridian zenith distances of two stars, one of which culminates north and the other south of the zenith, within a short interval of time of each other.

Let λ be the latitude, δ and z the declination and zenith distance of the southern star, δ' and z' those of the northern star. Then, since the latitude is the same as the zenith distance of the equator, we have:

$$\lambda = \delta + z$$
$$\lambda = \delta' - z'$$

and adding, $2\lambda = \delta + \delta' + z - z'$

Therefore, if δ and δ' are known exactly we can find the latitude from the *difference* of z and z' without knowing their actual values. Moreover, if z and z' are nearly the same the refractions will nearly neutralize each other, and we shall only have to take into account the difference of the refractions at the two altitudes.

The instrument is practically a telescope about 45 inches focal length, attached to a vertical axis round which it revolves, having been first clamped at a certain angle of

elevation. The latitude must be known approximately, and a pair of stars selected which are of so nearly the same meridian zenith distance at that latitude that they will both pass within the field of view of the telescope without our having to alter its angle of elevation. As a rule, z and z' must not differ by more than $50'$ at the most. If the axis is truly vertical and the telescope remains at the same vertical angle at the observation of both stars, then it is plain that the difference of z and z' may be read by a micrometer in the eye piece.

It is usual to observe only stars which pass within 25 degrees of the zenith. The telescope has a long diagonal eye piece with a micrometer in its focus, and the micrometer wire is at right angles to the meridian. There is a very delicate level attached to the telescope, and a vertical arc which serves as a finder. By reading this level at each observation we can detect and allow for any change in the angle of elevation of the telescope.

The above is the merest outline of the principle of the instrument, and reference must be made to other works for the details of its construction. The method of using it is this: The latitude being already approximately known, a pair of stars is found from a star catalogue, both of which will pass within the field of view without altering the elevation, and which have nearly the same right ascension. The reason for this is that their transit may take place within so short an interval of time that the state of the instrument may remain unchanged; but a sufficient interval must be allowed for reading the micrometer and level and reversing in azimuth; say, not less than one minute or more than twenty. The meridian line must have been previously ascertained by transits of known stars, or otherwise, and the chronometer time calculated at which each of the stars will culminate. The telescope having been brought into the meridian, ready for the star which culminates first, and set for the mean

altitude of the two stars, the observer watches the first star enter the field of view, and bisects it with the micrometer wire at the calculated instant of its transit. He then reads the level and micrometer, revolves the telescope in azimuth through 180°, and observes the second star in the same manner. If, after the revolution, the level is much out, it must be relevelled by a tangent screw provided for the purpose, which does not alter the connection between the telescope and the level, but moves them together.

The micrometer has a single screw and one or more moveable wires, and reads to as little as 0″.05. It usually has several fixed vertical wires, so that the instrument may be used for transits.

Besides the vertical arc for setting the telescope at the required elevation, there is a horizontal circle with two stops for getting it in the plane of the meridian, and also a striding level for setting the axis vertical.

In the field the instrument is mounted like the transit telescope.

This method of finding the latitude is known as Talcott's, having been invented by Captain Talcott of the U. S. Engineers. Its defects are that it is often difficult to obtain a sufficient number of suitable pairs of stars, of which the declinations are accurately known. As a rule we have to use the smaller stars, whose places are not very well known, and must therefore observe a large number of pairs to eliminate errors.

TO FIND THE CORRECTED LATITUDE.

Let m be the micrometer reading (in arc) for the southern star, m_0 the same for any point in the field assumed as the micrometer zero, and z_0 the apparent zenith distance represented by m_0 when the level reading is zero. Suppose, also, that the micrometer readings increase as the zenith distances decrease. Then, if the level reading were zero, the star's apparent zenith distance would be

$$z_0+m_0-m$$

Let l be the equivalent in arc of the level reading, position when the reading of the north end of the level is the greater. Let r be the refraction. Then the true zenith distance of the southern star, or z, is:

$$z_0+m_0-m+l+r$$

The quantity z_0+m_0 is constant so long as the relation of the level and telescope is not changed. We have, therefore, for the northern star,

$$z'=z_0+m_0-m'-l'+r'$$

Hence

$$z-z'=m'-m+l'+l+r-r'$$

and the equation for the latitude previously given will become:

$$\lambda=\tfrac{1}{2}(\delta+\delta')+\tfrac{1}{2}(m'-m)+\tfrac{1}{2}(l'+l)+\tfrac{1}{2}(r-r')$$

TO FIND THE CORRECTION FOR LEVEL.

Calling the readings of the north and south ends of the bubble n and s, and the inclinations at the observations of the north and south stars, expressed in divisions of the level, L′ and L, we shall have

$$L'=\frac{n'-s'}{2} \qquad L=\frac{n-s}{2}$$

and if D is the value of a division of the level in seconds of arc, we have

$$l'=L'\,D \qquad l=L\,D$$

and the correction for the level will be

$$\tfrac{1}{2}(l'+l)=\tfrac{1}{2}(L'+L)\,D=\frac{n'+n-(s'+s)}{4}\,D$$

TO FIND THE VALUE OF A DIVISION OF THE LEVEL.

Turn the telescope on a well-defined distant mark. Set the level to an extreme reading L, bisect it by the micrometer wire, and let the micrometer reading be M. Now move the telescope and level together by the tangent

screw till the bubble gives a reading L′ near the other extreme, bisect the mark again by the wire, and let the micrometer reading be M′. The value of a division of the level in turns of the micrometer will be

$$d = \frac{M - M'}{L' - L}$$

and if R is the value in seconds of arc of a revolution of the micrometer, the value D of the level in seconds of arc will be

$$D = Rd$$

TO FIND THE VALUE OF A REVOLUTION OF THE MICROMETER SCREW.

This is best done by observations of a circumpolar star near its greatest elongation. We find its hour angle and zenith distance by the formulœ

$$\cos t = \cot \delta \tan \lambda$$
$$\cos z = \operatorname{cosec} \delta \sin \lambda$$

Whence, knowing the star's R. A. and the chronometer error, we find the chronometer time of the greatest elongation. Set the telescope for the zenith distance z, direct it upon the star 20 or 30 minutes before the time of greatest elongation, and bisect it with the micrometer wire; note the time of bisection, and micrometer and level readings. As the star moves vertically repeat this process as often as possible while it is in the field of view. Let t_1, t_2, t_3, &c., be the noted chronometer times of bisection, m_1, m_2, m_3, &c., the corresponding micrometer readings, m the micrometer reading at the instant of greatest elongation (t), and i_1, i_2, i_3, &c., the required angular distances; then the latter are found from the equations

$$\sin i_1 = \sin (t - t_1) \cos \delta$$
$$\sin i_2 = \sin (t - t_2) \cos \delta$$

If R is the value, in seconds, of a revolution of the micrometer, and if the level has remained constant, we have, since $(m - m_1)$ is the number of turns given to the

screw to move the thread through the angular distance i_1

$$(m-m_1)\ R=i_1$$
$$\text{Also } (m-m_2)\ R=i_2$$

Therefore, subtracting

$$(m_2-m_1)\ R=i_1-i_2$$
$$\text{or } R=\frac{i_1-i_2}{m_2-m_1}$$

To correct for any change in the level reading, let l_1 and l_2 be the level readings corresponding to m_1 and m_2; then (l_2-l_1) D is the change required. The angular value of D is unknown; but, since D$=d$R, the correction to be applied to (i_1-i_2) is $(l_2-l_1)\ d$R; and

$$(m_2-m_1)\ R=i_1-i_2 \pm (l_2-l_1)\ dR$$
$$\text{or } R=\frac{i_1-i_2}{(m_2-m_1) \mp (l_2-l_1)\ d}$$

A value of R is thus obtained for each of the observations, and the mean of the results taken. This mean has then to be corrected for refraction, thus: From the tables find the change in refraction for 1′ at the zenith distance z. Let this change be dr; then R dr will be the correction to be subtracted from R.

REDUCTION TO THE MERIDIAN.

If a star has not been observed exactly on the meridian it may be taken when off it, and the observation reduced. The following is one method of doing this. Keeping the instrument clamped in the meridian, the star is observed at a certain distance from the middle vertical thread and the time noted. This will give its hour angle, and if we denote this by t (in seconds of time) the reduction is obtained by the formula

$$\tfrac{1}{4}\ (15\ t)^2 \sin 1'' \sin 2\,\delta$$

This is to be added to the observed zenith distance of a southern star, or subtracted from that of a northern one, and, in either case, half of it is to be added to the latitude.

REFRACTION.

When the zenith distances are small the refraction varies as the tangent of the zenith distance.

$$\text{Let } r = a \tan z$$
$$r' = a \tan z'$$
$$\text{Then } r - r' = a\,(\tan z - \tan z')$$
$$= a\,\frac{\sin\,(z - z')}{\cos z \cos z'}$$
$$= (z - z')\frac{a \sin 1'}{\cos^2 z'}, \text{ nearly}$$

a may be taken as $57''.7$, and the difference of the micrometer readings used for $(z - z')$

THE PORTABLE TRANSIT INSTRUMENT AS A ZENITH TELESCOPE.

If the portable transit telescope has a micrometer added to it, and the level of the finder circle is made sufficiently delicate, it may be used as a zenith telescope, reversing the instrument in its Ys between the observations.

Note—The catalogues give the *mean* places of the stars. The *apparent* places are those which have to be used, and must therefore be determined.

CHAPTER IX.

ADDITIONAL METHODS OF FINDING THE LATITUDE.

TO FIND THE LATITUDE BY A SINGLE ALTITUDE TAKEN AT A KNOWN TIME.

Here we have in the triangle P Z S the hour angle P, the side Z S (90°—the objects altitude), and P S the polar distance. From these data we have to find P Z. From S draw S M perpendicular to P Z produced. Let δ be the declination, φ the latitude, and α the altitude. In the triangle P M S we have:

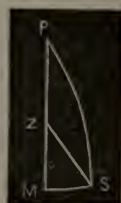


Fig. 24.

$$\cos P = \tan PM \cot PS = \tan PM \tan \delta \qquad (1)$$

$$MZ = PM - PZ = PM + \varphi - 90^\circ$$

Also, since $\cos PM = \frac{\cos PS}{\cos MS}$ & $\cos ZM = \frac{\cos ZS}{\cos MS}$

$$\cos PM : \cos ZM :: \cos PS : \cos ZS$$

$$\text{or } \cos PM : \sin (PM + \varphi) :: \sin \delta : \sin \alpha$$

Therefore

$$\sin (PM + \varphi) = \frac{\sin \alpha \cos PM}{\sin \delta} \qquad (2)$$

Equation (1) gives P M and (2) gives P M $+ \varphi$

In this method, if the star is observed when far from the meridian a small error in the hour angle produces a large error in the computed value of the latitude. The altitude should therefore be taken when the object is near the meridian.

TO FIND THE LATITUDE BY OBSERVATIONS OF THE POLE STAR OUT OF THE MERIDIAN.

If p be the polar distance of the pole star in circular measure p^3 is a very small quantity.

Fig. 25.

Let P be the pole, Z the zenith, and S the star at an hour angle h or S P Z. Draw S N at right angles to P Z and take Z M equal to Z S. Let P N be denoted by x, M N by y, S P by p, the star's altitude by a, and the latitude by l. Then

$$\text{P}\,Z=Z\,\text{M}+\text{M P}=Z\,\text{S}+\text{P N}-\text{N M}$$

$$\text{or}\quad 90-l=90-a+x-y$$

$$\therefore\ l=a-x+y$$

We have to find x and y.

(1) From the right-angled triangle S P N we have

$$\cos \text{S P N}=\tan \text{P N}\cot \text{P S}$$

$$\therefore\ \tan x=\tan p\cos h$$

$$\text{or, approximately, } x=p\cos h$$

(2) Denoting S N by q we have from the same triangle

$$\text{Sin S N}=\sin \text{S P}\sin \text{S P N}$$

$$\text{or } \sin q=\sin p\sin h$$

$$\therefore\ \text{approximately, } q=p\sin h.$$

(3) In the right-angled triangle S N Z we have

$$\cos Z\,\text{S}=\cos \text{S N}\cos Z\,\text{N}$$

$$\therefore\ \sin a=\cos q\sin(a+y)$$

$$\text{or } \sin(a+y)=\frac{\sin a}{\cos q}$$

or approximately

$$\sin a+\overset{y.}{\cancel{q}}\cos a=\frac{\sin a}{1-\frac{1}{2}q^2}$$

$$=\sin a\left(1+\tfrac{1}{2}q^2\right)$$

$$y\cos a=\tfrac{1}{2}q^2\sin a$$

$$\text{or } y=\tfrac{1}{2}q^2\tan a$$

$$=\tfrac{1}{2}p^2\sin^2 h\tan a$$

Hence, in circular measure

$$l=a-p\cos h+\tfrac{1}{2}p^2\sin^2 h\tan a$$

or in sexagesimal measure

$$l=a-p\cos h+\tfrac{1}{2}p^2\sin 1''\sin^2 h\tan a$$

This is the method given in the explanations at the end of the Nautical Almanac. To find the latitude we have only to take an altitude of Polaris, note the time (which will give us the sidereal time), and apply certain corrections as directed in the Almanac.

FINDING THE LATITUDE BY CIRCUM-MERIDIAN ALTITUDES.

When the latitude has been found by a single meridian altitude the result is only approximately true. It may, however, be obtained with great exactness by taking a number of altitudes of the sun or a star when within about a quarter of an hour of the meridian on either side of it. The altitudes may be taken with the sextant, reflecting circle, or theodolite, and the observations should follow each other quickly, and at about equal intervals of time.

The watch error must be exactly known, and the time of each altitude noted. The mean of the altitudes is taken, but the hour angle for each must be obtained separately. In the case of the sun this is done by correcting the observed times for watch error and subtracting them from the mean time of apparent noon. If a star is used the mean time corresponding to its R. A. will, of course, give the hour angles—

The formula is

$$\text{Latitude}=90^\circ-a\pm d-x''$$

Where a is the mean of the altitudes, d the declination of the object (negative if south), and x'' a quantity equal to

$$\frac{2\sin^2\frac{h}{2}}{\sin 1''}\times\text{cos. approx. lat.}\times\text{cos. dec'n.}\times\text{sec. alt.};$$ h being the hour angle.

To prove the formula
$l=90°-a\pm d-x''$, when the latitude is approximately known,

$$\text{and } x''=\frac{2\sin^2\frac{h}{2}}{\sin 1''}\times\frac{\cos l\cos d}{\cos a}$$

Let P be the pole, Z the zenith, and S the sun or star near the meridian.

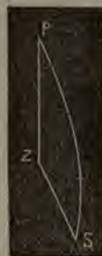


Fig. 36.

Let a be the star's altitude, h its hour angle, and d its declination.

Let $a+x$ be the star's meridian altitude. Then $a+x\mp d=90-l$

We have now to determine the small quantity x.

Now, sin PZ sin PS cos ZPS=cos ZS—cos PZ cos PS
~~cos PS~~ or $\cos l\cos d\cos h=\sin a-\sin l\sin d$

$$\therefore \cos l\cos d\,(1-\cos h)=-\sin a+\cos(l-d)$$
$$=-\sin a+\sin(a+x)$$

$$\therefore 2\cos l\cos d\sin^2\frac{h}{2}=2\sin\frac{x}{2}\cos\left(a+\frac{x}{2}\right)$$

Therefore, approximately

$$x''=\frac{2\sin^2\frac{h}{2}}{\sin 1''}\times\frac{\cos l\cos d}{\cos a}$$

d is, of course, negative if south.

The value of the expression $\dfrac{2\sin^2\frac{h}{2}}{\sin 1''}$ (known as the "reduction to the meridian") is found for each hour angle from a table, and the mean of all the values taken to calculate x'' with.

Example—On a certain date, at a place the latitude of which had been approximately ascertained to be 29° 52′ 0″, the mean of ten altitudes of the sun's lower limb, observed with a powerful theodolite, was 39° 59′ 20″. This,

when corrected for refraction, parallax, and semi-diameter, gave $40° 14' 31''.55$ as the true mean altitude of the sun's centre. The sun's declination was $19° 53' 45''.8$ south. The mean of the values of the reduction for the observed hour angles, as taken from the table, was $16''.26$, and the calculated value of x was $17''.36$.

	90°	0′	0″
Altitude	40	14	31.55
	49	45	28.45
Declination...	19	53	45.80
	29	51	42.65
			17.36
Latitude =	29	51	25.29

Strictly speaking, a further correction ought to be made for the change in the sun's declination during the observations.

In the case of a star we must add 0.0023715 to the log. of x'' to correct the hour angle for the difference between the sidereal and mean time intervals; for the star moves faster than the sun, and therefore gives a larger hour angle for the same time.

Additional accuracy is obtained by taking half the observations east of the meridian, and half west of it, the intervals of time between the successive observations being made as nearly equal as possible. The hour angle changes its sign after the meridian passage of the object.

CHAPTER X.

INTERPOLATION.

METHODS OF FINDING THE LONGITUDE.

In taking out variable quantities from the Nautical Almanac it is necessary to interpolate for the local time and longitude of the place of observation, since the data given are for Greenwich time.

If the rate of change of the variable quantities is itself variable we must allow for it if we wish to obtain a very accurate result.

As an instance: we wish to find with great exactness the sun's declination at apparent noon on the 2d January of a certain year, at a place in longitude 60° or 4h. west.

For Greenwich mean noon we find in the Almanac

Date.	Sun's declination.	Variation in 1 hour.
2nd January	22° 57′ 16″.2	13″.21
3rd "	22° 51′ 45″.4	14″.35

Now, at apparent noon at the place it will be 4 P.M. apparent time at Greenwich, and we take the variation at 2 P.M. Greenwich as the *average* variation for those 4 hours.

This variation is 13″.305, which multiplied by 4 gives

53″.22 to be subtracted from the declination of 2d January—

$$\begin{array}{r} 22^\circ\ 57'\ 16''.2 \\ 53''.22 \\ \hline 22\ \ 56\ \ 22.98 \end{array} = \text{required dec'n.}$$

$$\begin{array}{r} 14.35 \\ 13.21 \\ \hline 12)\ 1.14 \\ \hline .095 \\ 13.21 \\ \hline 13.305 \\ 4 \\ \hline 53.22 \end{array} \quad 13.305 = \text{Variation at 2 P.M}$$

INTERPOLATION BY SECOND DIFFERENCES.

The differences between the successive values of the quantities given in the Nautical Almanac as functions of the time are called the *first differences;* the differences between these successive differences are called *second differences;* the differences of the second differences are *third differences*, and so on. In simple interpolation we assume the function to vary uniformly; that is, that the first difference is constant, and therefore that there is no second difference. If this is not the case simple interpolation will give an incorrect result, and we must resort to interpolation by second differences, in which we take into account the variation in the first difference, but assume its variation to be constant and that there is no third difference.

The formula employed is

$$f(a+k)=f(a)+\mathrm{A}\,k+\mathrm{B}\,k^2$$

where A is half the sum of two consecutive first differences and B is half their difference. It is thus derived:

We have by Taylor's Theorem

$$f(x+h)=f(x)+\mathrm{A}h+\mathrm{B}h^2+\&\text{c.} \qquad \text{(A)}$$

and if h is small compared with x the successive terms of the series grow rapidly less.

Suppose $a-1$, a, and $a+1$ to be three successive arguments of a table constructed from $f(x)$ in which it is assumed that a is many times greater than 1. Then, from the table we know $f(a-1)$, $f(a)$, and $f(a+1)$, and therefore we know the differences $f(a)-f(a-1)$, and $f(a+1)-f(a)$, which we may designate by $\triangle$ and $\triangle'$ respectively.

Knowing that third differences can be neglected we can obtain the value of $f(a+k)$, where k is less than 1, as follows:

From (A) we have, if we neglect higher powers,

$$f(a-1)=f(a)-A+B \qquad (1)$$
$$f(a)=f(a) \qquad (2)$$
$$f(a+1)=f(a)+A+B \qquad (3)$$
$$f(a+k)=f(a)+Ak+Bk^2 \qquad (4)$$

Subtracting equation (1) from (2) we get

$$\triangle=A-B$$

and subtracting (2) from (3) $\triangle'=A+B$

$$\therefore\ A=\tfrac{1}{2}(\triangle'+\triangle)$$
$$B=\tfrac{1}{2}(\triangle'-\triangle)$$

$\therefore$ substituting in (4)

$$f(a+k)=f(a)+\tfrac{1}{2}(\triangle'+\triangle)k+\tfrac{1}{2}(\triangle'-\triangle)k^2$$

The signs of A and B must be carefully noted. If the functions are decreasing the first differences are negative, and if the first differences are decreasing the second differences are negative.

The method can be better understood from an example or two.

Ex. 1.—Given the logs. of 365, 366, and 367 to 7 places of decimals to determine log. 366.4.

Numbers.	*Log.*	*1st Differ'ce.*	*2d Differ'ce.*
365	5622929	11882	—32
366	5634811	11850	
367	5646661		

Here k is $\frac{4}{10}$, $A = 11866$, and $B = -16$.

5634811	$11866 \times \frac{4}{10}$
4746	4
3639557	47464
3	$-\frac{32}{2} \times (\frac{4}{10})^2 = -3$, nearly.
3639554	

The tables give the log. as 3639555.

If the second difference had been neglected—*i.e.*, if we had worked by simple interpolation, the result would have been 5639551.

Ex. 2.—Given the log. cosines of 89° 32′, 89° 33′, and 89° 34, to find log. cos. 89° 33′ 15″.

	1st Difference.	2nd Difference.
Log. cos. 89 32=7.9108793		
	—157939	
Log. cos. 89 33=7.8950854		—5962
	—163901	
Log. cos. 89 34=7.8786953		

Here we have to *subtract* $\frac{15}{60}\times$half the sum of the 1st differences, and $(\frac{15}{60})^2\times$half the second difference, or 40416 in all;

$\therefore$ log. cos. 89° 33′ 15″=7.8910438.

TO FIND THE GREENWICH TIME CORRESPONDING TO A GIVEN RIGHT ASCENSION OF THE MOON ON A GIVEN DAY.

Let T′=the Greenwich time corresponding to the given right ascension a'

" T=the Greenwich hour preceding T′ and corresponding to the right ascension a

" $\triangle a$=the difference of R. A. in one minute at the time T.

Then we shall have, approximately,

$$T'-T=\frac{a'-a}{\triangle a}$$

To correct for second differences we have now only to find the difference of R. A. for one minute at the middle instant of the interval T′—T. Call this Δ', and we shall have

$$T'-T=\frac{a'-a}{\Delta'\,a}$$

T and T′ are in minutes.

INTERPOLATION BY DIFFERENCES OF ANY ORDER.

If it is required to find the intermediate values of a function with greater exactness than can be done by interpolation by second differences we can use any number of differences.

Let	T, T+w, T+2 w, &c.,	be the arguments.
"	F, F′, F″, &c.,	" the functions.
"	a, a', a'' &c.,	the 1st differences.
"	b, b' b'' &c.,	" 2nd "
	&c.,	&c.,

So that $F'-F=a$, $a'-a=b$, and so on.

Now, if $F^{(n)}$ is the function corresponding to the argument T+n w we have

$$F^{(n)}=F+n\,a+\frac{n(n-1)}{1.2}b+\frac{n(n-1)\,(n-2)}{1.2.3}+\&c. \quad (a)$$

If n be taken successively equal to 0,1,2, &c., we shall obtain the functions F, F′, F″, &c., and intermediate values are found by using fractional values of n. To find the proper value of n in each case let T+t denote the value of the argument for which we wish to interpolate a value of the function ; then

$n\,w=t$, and $n=\dfrac{t}{w}$; that is, n is the value of t reduced to a fraction of the interval w.

Ex.—Suppose the moon's R. A. had been given in the Almanac for every 12th hour, as follows :

	Moon's R. A.	*1st Diff.*	*2nd Diff.*	*3d Diff.*	*4thDiff*	*5thDif*
Mar. 5, oh	21h. 58m 28s .39	+28m. 47s.04				
" 5, 12h	22 27 15 .43	28 10 .07	—36s.97			
" 6, oh	22 55 25 .50	27 37 .89	32 .18	+4s.79	+1s .74	—0s.66
" 6, 12h	23 23 3 .39	27 12 .24	25 .65	6 .53	1 .08	
" 7, oh	23 50 15 .63	26 54 .20	18 .04	7 .61		
" 7, 12h	0 17 9 .83					

required the moon's R. A. for March 5, 6h.

Here T=March 5, 0_h, $t=6^h$, $w=12^h$, $n=\frac{6}{12}=\frac{1}{2}$; and if we denote the co-efficients of a, b, c, &c. in (a) by A, B, C, &c., we have

$$F=21^h\ 58^m\ 28^s.39$$

$$a=+28^m\ 47^s.04,\ A=n \qquad =\tfrac{1}{2},\ Aa=+14^m\ 23^s.52$$

$$b=-\ 36^s.97,\ B=A\,\frac{n-1}{2} \qquad =-\tfrac{1}{8},\ Bb=+\ 4^s.62$$

$$c=+\ 4^s.79,\ C=B\,\frac{n-2}{3} \qquad =+\tfrac{1}{16},\ Cc=+\ 0^s.30$$

$$d=+\ 1^s.74,\ D=C\,\frac{n-3}{4} \qquad =-\tfrac{5}{128},\ Dd=-\ 0^s.07$$

$$e=-\ 0^s.66,\ E=D\,\frac{n-4}{5} \qquad =+\tfrac{7}{256},\ Ee=-\ 0^s.02$$

Moon's R. A. on March 5, 6^h, or $F^{(\frac{1}{2})} \quad =22^h\ 12^m.\ 56_s.74$

TO FIND THE LONGITUDE BY TRANSITS OF MOON-CULMINATING STARS.

This is a simple and easy way of finding the longitude when the meridian line is known, though not a very accurate one; for an error of one second in an observed transit may throw the longitude out as much as half a minute in time, or $7\frac{1}{2}$ minutes in arc. It is, however, a method that may be useful to a surveyor, since all he requires for it is a transit theodolite and an ordinary watch. Of course, a portable transit instrument is to be preferred, if available.

The instrument is set up in the plane of the meridian,

and the *interval of time* noted between the transits of the bright limb of the moon and of certain stars, the right ascension and declination of which are nearly the same as that of the moon at the time. It is not necessary to know either Greenwich or local time, but the rate of the watch should be taken into account. The instants of transit are noted, and the interval of time between them is reduced from mean to sidereal time.

In the Nautical Almanac are given, for every day of the year, the sidereal times of transit at Greenwich of the moon and of certain suitable stars, called "moon-culminating" stars; also the rate of change per hour (at the time of transit) of the moon's R. A. As the moon moves rapidly through the stars from west to east, it is evident that at a station not on the meridian of Greenwich the interval between the two transits will be different to that at Greenwich; and, the moon's rate of motion per hour being known, a simple proportion will (if the station is near the meridian of Greenwich) give the difference of time between the station and Greenwich, and thence the longitude. If the station is far from the meridian of Greenwich a correction will have to be made for the change in the rate of change of the moon's R. A. The rate of change at the time of transit is found from the Nautical Almanac by interpolation by second differences, and the mean of the rates of change at Greenwich and at the station is taken as the rate for the whole interval of time between the transits.

An example will best illustrate the method :—

At Kingston, Canada, on the 24th February, 1882, the transits of the star v Tauri and of the moon's bright limb were observed at 6h. 0m. 23s., and 6h. 1m. 9s. respectively, mean time. Difference, 46 seconds, or 46s.12 in sidereal units.

Greenwich Transits	v Tauri...4h.	19m.	16s.62	
	Moon I...4	7	57 .44	

Difference in sidereal time=	11m. 19s.18
Add interval at Kingston=	46 .12

Total change of moon's R.A= 12m 5s.3=725s.3

By interpolation by second differences the variation of the moon's R.A. per hour at Kingston at the time of transit was found to be.................... 142s .23

At Greenwich it was................. 142 .68

2)284 .91

Mean rate of variation.................142.455

$$\frac{725\ .3}{142.45} \times 1\text{h.} = 5\text{h.}0916$$

5h. 5m. 29s.76 west longitude.

It should be noted that in this case the moon was west of the star at transit at Greenwich and east of it at Kingston, having passed it in the interval.

The following is a specimen of the part of the Nautical Almanac relating to moon-culminating stars.

MOON-CULMINATING STARS, 1880.

AT TRANSIT AT GREENWICH.

Month and Day.	Name.	Magnitude.	*Apparent* Right Ascension.	Cor. of *'s R.A. for foll. Day.	Var. of Moon's R.A. in 1 hour of Long.	Sid. Time of Semid pass'g Merid	*Apparent* Declination.	Var. of Moon's Dec. in 1 hour of Long.	Semidiameter.	Hor. Par.
			h m s	s	s	s	° ′ ″	″	′ ″	′ ″
July 15	Moon I. U.	8.22	13 42 56.22		140.57	69.60	S. 15 57 21.2	— 695.8	15 58.5	58 31.8
	Moon I. L.	-	14 11 35.99		146.13	70.46	18 10 13.1	— 630.2	16 4.9	58 55.3
	B.A.C. 4673	6½	13 55 59.40	— .02			19 14			
	B.A.C. 4722	6	14 8 50.88	— .01			17 38			

FINDING THE LONGITUDE BY LUNAR DISTANCES.

This method is an important one to the travelling astronomer, and to the navigator whose chronometer has gone wrong. The instrument used is the sextant or some other reflecting one, and the observation is a very simple one. An error of 30″ in reading the angle, causes, however, an error in longitude of about a quarter of a degree.

The moon moves amongst the stars from west to east at the rate of about 13° a day. Its angular distance from the sun or certain stars may therefore be taken as an indication of Greenwich mean time at any instant—the moon being in fact made use of as a clock in the sky to show Greenwich mean time at the instant of observation. The local mean time being also supposed to be known, we have the requisite data for determining the longitude of a station.

In the Nautical Almanac are given for every 3d hour of G.M.T. the angular distances of the apparent *centre* of the moon from the sun, the larger planets, and certain stars, as they would appear from the centre of the earth. When a lunar distance has been observed it has to be reduced to the centre of the earth by clearing it of the effects of parallax and refraction, and the data ~~numbers~~ in the Nautical Almanac give the exact Greenwich mean time at which the objects would have the same distance.

It is to be noted that, though the combined effect of parallax and refraction increases the apparent altitude of the sun or a star, in the case of the moon, owing to its nearness to the earth, the parallax is greater than the refraction, and the altitude is lessened.

Three observations are required—one of the lunar distance, one of the moon's altitude, and one of the other object's altitude. The altitudes need not be observed with the same care as the distance. The clock time of the observations must also be noted. The sextant is the instrument generally used. All the observations can be taken by one observer, but it is better to have three or four. If one of the objects is at a proper distance from the meridian the local mean time can be inferred from its altitude. If it is too near the meridian the watch error must be found by an altitude taken either before or after the lunar observation.

Four or five sets of observations should be made and written down in their proper order:

	Time by watch.	Alt. of star.	Alt. of moon's lower limb.	Dist. of moon's far limb
1st obs'n	....	....	....	....
2nd "	....	....	....	...
3rd "	....	....	....	....
4th "	....	....	...	....
	4)			Totals.
Mean	....	....	....	...

If there is only one observer it is best to take the observations in the following order, noting the time by a watch. 1st, alt. of sun, star or planet; 2d, alt. of moon; 3d, any odd number of distances; 4th, alt. of moon; 5th, alt. of sun, star, or planet. Take the mean of the distances and of their times. Then reduce the altitudes to the mean of the times; *i.e.*, form the proportion—difference of times of altitudes : diff. of alts. :: diff. between time of 1st alt. and mean of the times : a fourth number which is to be added to or subtracted from 1st alt. according as it is increasing or diminishing. This will give the altitudes reduced to the mean of the times, or corresponding to that mean.

The altitudes of moon and star must be corrected as usual, and the augmented semi-diameter of the moon added to the distance to give the distance of its centre. The lunar distance has then to be cleared of the effects of parallax and refraction.

TO DETERMINE THE LUNAR DISTANCE CLEARED OF PARALLAX AND REFRACTION.

Let Z be the observer's zenith, Zm and Zs the vertical circles in which the moon and star are situated at the instant of observation. Let m and s be their observed places, M and S their places after correction for parallax and refraction: then Zm, Zs, and ms are found by observation, and Z M and ZS are obtained by correcting the observations. The ob-

Fig. 27.

ject of the calculation is to determine M S.

Now, as the angle Z is common to the triangles mZs and M Z S, we can find Z from the triangle mZs in which all the sides are known. Next, in triangle M Z S there are known M Z, Z S, and the included angle Z, from which M S can be found. M S is the cleared lunar distance. The numerical work of this process is tedious.

The cleared distance having been obtained we proceed in accordance with the rules given in the N.A.

The Greenwich mean time corresponding to the cleared distance can be found either by a simple proportion or by proportional logs.

It admits of proof that if D is the moon's semi-diameter as seen from the centre of the earth (given in N.A.), D′ its semi-diameter as seen by a spectator in whose zenith it is, D″ its semi-diameter as seen at a point where its altitude is a, then

$$D'' - D = (D' - D) \sin a, \text{ very nearly.}$$

For details of the methods of finding differences of longitude by the transportation of chronometers, and by the electric telegraph, *vide* Chauvenet or Loomis.

CHAPTER XI.

MISCELLANEOUS.

TO FIND THE AMPLITUDE AND HOUR ANGLE OF A GIVEN HEAVENLY BODY WHEN ON THE HORIZON.

The *amplitude* is the angle that the plane of the vertical circle through an object makes with the plane of the prime vertical.

Fig. 28.

Let N S E W be the north, south, east, and west points of the horizon respectively; P the pole; and H the heavenly body. Suppose H to be between N and W. Join P H.

Here W H is the amplitude (a), and in the triangle H P N we have NP the latitude (φ), H P the object's polar distance ($90°—\delta$), and H N P a right angle. Also, if t is the hour angle, H P N$=180°—t$, and N H$=90°—a$. Hence:

$$\left.\begin{aligned} \sin a &= \sec\varphi \sin\delta \\ \cos t &= -\tan\varphi \tan\delta \end{aligned}\right\}$$

From the second of these equations we can calculate the time at which the heavenly body rises and sets.

TO FIND THE EQUATORIAL HORIZONTAL PARALLAX OF A HEAVENLY BODY AT A GIVEN DISTANCE FROM THE CENTRE OF THE EARTH.

Referring to the figure in the next article, if A is the observer's position H′ will be the apparent position of the heavenly body, and if C be the centre of the earth the equatorial horizontal parallax will be the angle H′. Designating A C by r, A H′ or C H by d, and the parallax by p, we have

$$\sin p = \frac{r}{d}$$

TO FIND THE PARALLAX IN ALTITUDE, THE EARTH BEING REGARDED AS A SPHERE.

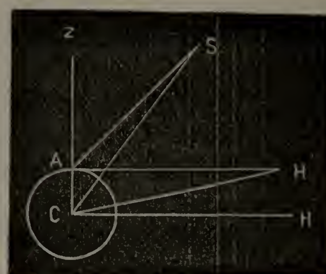


Fig 29.

In Fig. 29 A is the observer's position, *Z* the zenith, C H the rational horizon, A H′ the sensible horizon, and S the heavenly body. Let p be the horizontal parallax (H′), p' the parallax in altitude (S), h the altitude (S A H′), and d the distance of the heavenly body (S C). From the triangle S A C we have

$$\frac{\sin S}{\sin Z A S} = \frac{\sin S}{\sin S A C} = \frac{A C}{S C}$$

$$= \frac{r}{d} = \sin p$$

or $\sin p' = \cos h \sin p$

The angles p and p' being (except in the case of the moon) very small, we may substitute them for their sines, and the equation becomes

$$p' = p \cos h$$

STAR CATALOGUES.

If we want to find the position of a star not included amongst the small number (197) given in the Nautical Almanac we must refer to a *star catalogue*. In these

catalogues the stars are arranged in the order of their right ascensions, with the data for finding their apparent right ascensions and declinations at any given date. These co-ordinates are always changing. 1st. by precession, nutation, and aberration, which cause only apparent changes of position; 2ndly, by the proper motions of the stars themselves amongst each other. In the catalogues the stars are referred to a mean equator and a mean equinox at some assumed epoch. The place of a star so referred is called its *mean* place at that time; that of a star referred to the true equator and true equinox its *true* place; and that in which the star appears to the observer in motion its *apparent* place. The mean place at any time can be found from that of the catalogue by applying the precession and the proper motion for the time that has elapsed since the epoch of the catalogue; the true place will then be found by correcting the mean place for nutation; and, lastly, the apparent place is found by correcting the true place for aberration.

The most noteworthy star catalogues are the British Association Catalogue (B. A. C.) containing 8,377 stars, the Greenwich catalogues, Lalande's, containing nearly 50,000, Struve's, Argelander's, &c., &c.

DIFFERENTIAL VARIATIONS OF CO-ORDINATES.

It is often necessary in practical astronomy to determine what effect given variations of the data will produce in the quantities computed from them. If the variations are very small the simpler differential equations may be used. The most useful differential formulæ, as regards spherical triangles, are deduced as follows:

We have the fundamental equations:

$$\left.\begin{aligned}
&\cos a = \cos b \cos c + \sin b \sin c \cos A \\
&\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A \\
&\sin a \sin B = \sin b \sin A \\
&\sin a \cos C = \sin b \cos c - \cos b \sin c \cos A \\
&\sin a \sin C = \sin c \sin A
\end{aligned}\right\} \quad (1)$$

Differentiating the first equation of this group and changing signs, we have

$$\sin a\, da = \sin b \cos c\, db + \cos b \sin c\, dc - \cos b \sin c \cos A\, db$$
$$- \sin b \cos c \cos A\, dc + \sin b \sin c \sin A\, d\,A$$
$$= (\sin b \cos c - \cos b \sin c \cos A)\, db +$$
$$(\cos b \sin c - \sin b \cos c \cos A)\, dc + \sin b \sin c \sin A\, d\,A$$
$$= \sin a \cos C\, db + \sin a \cos B\, dc + \sin b \sin c \sin A\, da$$

$$\text{or } da = \cos C\, db + \cos B\, dc + \sin b \sin c \frac{\sin A}{\sin a} d\,A$$
$$= \cos C\, db + \cos B\, dc + \sin b \sin C\, d\,A$$

$$\left.\begin{array}{l} \text{or } da - \cos C\, db - \cos B\, dc = \sin b \sin C\, dA \\ \quad\text{Similarly we obtain} \\ -\cos C\, da + db - \cos A\, dc = \sin c \sin A\, d\,B \\ -\cos B\, da - \cos A\, db + dc = \sin a \sin B\, d\,C \end{array}\right\} \quad (2)$$

From these, by eliminating da, we obtain:

$$\left.\begin{array}{l} \sin C\, db - \cos a \sin B\, dc = \sin b \cos C\, d\,A + \sin a\, d\,B \\ -\cos a \sin C\, db + \sin B\, dc = \sin a \cos B\, d\,A + \sin a\, d\,C \end{array}\right\} (3)$$

and by eliminating db from these:

$$\sin a \sin B\, dc = \cos b\, d\,A + \cos a\, d\,B + d\,C \qquad (4)$$

If we eliminate $d\,A$ from (3) we get

$$\cos b \sin C\, db - \cos c \sin B\, dc = \sin c \cos B\, d\,B$$
$$- \sin b \cos C\, d\,C$$

and, by dividing this equation by $\sin b \sin C$ or its equivalent $\sin c \sin B$, we have

$$\cot b\, db - \cot c\, dc = \cot B\, d\,B - \cot C\, d\,C \qquad (5)$$

As an example, take the astronomical triangle P Z S, and put

$$\begin{array}{ll} A = Z & a = 90^\circ - \delta \\ B = t & b = \zeta \\ C = q & c = 90^\circ - \varphi \end{array}$$

Then the first equations of (2) and (3) give

$$\left.\begin{array}{l} d\,\delta = -\cos q\, d\,\zeta - \sin q \sin \zeta\, d\,Z + \cos t\, d\,\varphi \\ \cos\delta\, d\,t = \sin q\, d\,\zeta - \cos q \sin \zeta\, d\,Z + \sin\delta \sin t\, d\,\varphi \end{array}\right\} (6)$$

which determine the errors $d\,\delta$ and $d\,t$ in the values of δ and t computed according to the formulæ

$$\left.\begin{aligned} \sin\delta &= \sin\varphi\cos\zeta + \cos\varphi\sin\zeta\cos Z \\ \cos\delta\cos t &= \cos\varphi\cos\zeta - \sin\varphi\sin\zeta\cos Z \\ \cos\delta\sin t &= \sin\zeta\sin Z \end{aligned}\right\} (7)$$

(which are derived directly from the fundamental equations (1)), when ζ, Z, and φ, are affected by the small errors $d\zeta$, $d\,Z$, and $d\,\varphi$ respectively.

In a similar manner we obtain

$$\left.\begin{aligned} d\,\zeta &= -\cos q\,d\,\delta + \sin q\cos\delta\,d\,t - \cos Z\,d\,\varphi \\ \sin\zeta\,d\,Z &= \sin q\,d\,\delta + \cos q\cos\delta\,d\,t - \cos\zeta\sin Z\,d\,\varphi \end{aligned}\right\} (8)$$

which determine the errors $d\,\zeta$ and $d\,Z$ in the values of ζ and Z computed by the formulœ (derived as above)

$$\left.\begin{aligned} \cos\zeta &= \sin\varphi\sin\delta + \cos\varphi\cos\delta\cos t \\ \sin\zeta\cos Z &= \cos\varphi\sin\delta - \sin\varphi\cos\delta\cos t \\ \sin\zeta\sin Z &= \cos\delta\sin t \end{aligned}\right\} (9)$$

when δ, t, and φ are affected by the small errors $d\,\delta$, $d\,t$, and $d\,\varphi$, respectively.

(It seems almost superfluous to point out that in these formulœ φ is the latitude, δ the star's declination, q the angle S, or parallactic angle, ζ the star's zenith distance, and t the star's hour angle.)

TO FIND THE CORRECTION FOR SMALL INEQUALITIES IN THE ALTITUDES WHEN FINDING THE TIME BY EQUAL ALTITUDES OF A FIXED STAR.

If from a change in the condition of the atmosphere the refraction is different at the two observations, equal apparent altitudes will not give equal true altitudes. To find the change $\triangle t$ in the hour angle t produced by a change $\triangle a$ in the altitude a we have only to differentiate the equation.

$$\text{Sin } a = \sin\varphi\sin\delta + \cos\varphi\cos\delta\cos t$$

Regarding φ and δ as constant: whence

$$\text{Cos } a\triangle a = -\cos\varphi\cos\delta\sin t\ 15\ \triangle t$$

where $\triangle a$ is in seconds of arc, and $\triangle t$ in seconds of time.

If the altitude at the west observation is the greater by

$\triangle a$ the hour angle is increased by $\triangle t$, and the middle time is increased by $\frac{\triangle t}{2}$, which is therefore the correction for the difference of altitudes. From the above equation its value is

$$\frac{\triangle a \cos a}{30 \cos \varphi \cos \delta \sin t}$$

If A is the azimuth of the object, we have

$$\text{Sin A} = \frac{\cos \delta \sin t}{\cos a}$$

and the formula may be written

$$\frac{\triangle a}{30 \cos \varphi \sin \text{A}}$$

which will be least when the denominator is greatest; that is, when A=90° or 270°, The star is therefore best observed on or near the prime vertical. Low altitudes are, however, ~~best~~ bad, owing to uncertainty in the refraction. If the star's declination is nearly equal to the latitude the interval between the observations will be short, which is an advantage, as the instrument will be less liable to change.

EFFECT OF ERRORS IN THE DATA UPON THE TIME COMPUTED FROM AN ALTITUDE.

We have, from the first differential equation (8), multiplying dt by 15 to reduce it to seconds of arc,

$$15 \sin q \cos \delta \, dt = d\zeta + \cos Z \, d\varphi + \cos q \, d\delta$$

If the zenith distance ~~above~~ alone is erroneous we have $d\varphi = 0$, and $d\delta = 0$, and

$$15 \, dt = \frac{d\zeta}{\sin q \cos \delta} = \frac{d\zeta}{\cos \varphi \sin \text{Z}}$$

from which it follows that a given error in the altitude will have the least effect upon the time when the object is on the prime vertical. Also, that these observations give the most accurate results when the place is on the equator, and the least accurate when at the poles.

By putting $d\,\zeta=0$, $d\,\delta=0$, and $\sin q\,\cos\delta=\cos\varphi\,\sin Z$, we have

$$15\,dt=\frac{d\varphi}{\cos\varphi\,\tan Z}$$

by which we see that an error in the latitude also produces the least effect when the star is on the prime vertical, or the observer on the equator. In the former case $\tan Z$ is infinite; therefore, if the latitude is uncertain, we can still get good results by observing stars near the prime vertical.

If $d\zeta=0$ and $d\varphi=0$ we have

$$15\,dt=\frac{d\delta}{\cos\delta\,\tan q}$$

Hence an error in the star's declination produces the least effect when the star is on the prime vertical (since $\tan q$ is a maximum when $\sin Z=1$), and that, of different stars, those near the equator are the best to observe.

In high latitudes it will often be necessary, in order to avoid low altitudes, to observe stars at a distance from the prime vertical. In this case small errors in the data will affect the clock correction. But if the star is observed on successive days on the same side of the meridian at about the same azimuth, the clock's *rate* will be accurately obtained, though its actual *error* will be uncertain.

If the same star is observed both east and west of the meridian, and at the same distance from it, constant errors $d\varphi$, $d\delta$, and $d\zeta$, will give the same value of dt, but with opposite signs. Hence one clock correction will be too large, and the other too small, and by the same amount, and their mean will be the true clock correction at the time of the star's meridian transit.

EFFECT OF ERRORS OF ZENITH DISTANCE, DECLINATION, AND TIME, UPON THE LATITUDE FOUND BY CIRCUM-MERIDIAN ALTITUDES.

The formula for finding the meridian zenith distance ζ' from a circum-meridian zenith distance ζ is

$$\zeta' = \zeta - A\, m$$

where $A = \dfrac{\cos\varphi \cos\delta}{\sin\zeta'}$ and $m = \dfrac{2 \sin^2 \frac{t}{2}}{\sin 1''}$

Differentiating this, and regarding A as constant, we have, since $d\varphi = d\zeta' + d\delta$

$$d\varphi = d\zeta + d\delta - \frac{A \sin t}{\sin 1''} 15\, d\,t$$

The errors $d\zeta$ and $d\delta$ affect the resulting latitude by their whole amount. The coefficient of dt has opposite signs for east and west hour angles; therefore, if observations are taken of a number of pairs of equal altitudes east and west of the meridian, a small constant error in the hour angles (or clock correction) will be eliminated in the mean. This result is practically attained by taking the same number of observations at each side of the meridian, and at nearly equal intervals of time.

An error in the assumed latitude which affects A is eliminated by repeating the computation with the latitude found by the first one.

THE PROBABLE ERROR.

To give an idea of what is meant by the term "*probable error*," we will suppose a rifleman to have fired a large number of bullets at a target at the same range and with equal care in aiming, and that, on examining the target, it is found that half of them have struck within three feet of the centre, and half outside that radius; then it may be assumed, ~~*a priori*,~~ that the chances of any one shot hitting within 3 feet of the centre are even—in other words, that it is an even chance whether or not the bullet will strike within that distance or not. And this distance may be taken as the probable error of any one shot.

Now, if we make a series of independent but equally careful measurements of a given quantity, such as an

angle or a base line, they will all differ more or less, the closeness of the agreement depending on the instruments employed and the care exercised; and the problem is to decide what value is to be taken as the most likely to be the correct one—in other words to have the smallest probable error.

If a_1, a_2, a_3, &c., are the different measurements, n their number, and m their mean, then $m = \frac{a_1 + a_2 + \&c. + a_n}{n}$;

and it follows as an arithmetical consequence that the algebraical sum of the quantities $(m - a_1)$, $(m - a_2)$, &c, $(m - a_n)$ will be equal to zero. These quantities are called the "*residuals.*" Another property of the mean is that *the sum of the squares of the residuals,* $(m - a_1)^2$, $(m - a_2)^2$, &c. is a minimum.

Now it admits of proof that the mean is that value, derived from the various measurements, which is likely to be nearest the truth. The value of the probable error of the mean is

$$\sqrt{\frac{(m-a_1)^2 + (m-a_2)^2 \;\&c., + (m-a_n)^2}{n}} \times 0.674489 \quad (1) \;*$$

And the probable error of any *one* measurement is the probable error of the mean multiplied by $\sqrt{n}$.

It must be borne in mind that by the probable error being taken as so much is meant that it is an even chance that the value taken is within that much of the truth without regard to sign. Thus, if l be the mean of a number of measurements of a base line, and 1 foot its probable error, it is an even chance that its real value lies between $l - 1$ and $l + 1$.

Instead of using the probable error of a result we often employ what is called its *weight*; a function which indicates the relative value to be assigned to the results as regards precision.

* Recent writers give $\sqrt{n(n-1)}$ as the denominator instead of n

The formula for the weight is

$$\frac{n^2}{2\{(m-a_1)^2+(m-a_2)^2+\&c.,+(m-a_n)^2\}} \qquad (2)$$

$$\therefore \text{Probable error} = \frac{0.476936}{\sqrt{\text{weight}}}$$

So that the weight varies inversely as the square of the probable error. From the property of the sum of the squares of the residuals being a minimum in the case of the mean, this method is often called the "method of least squares."

As a simple example of the calculation of the probable error we will take a side of a triangle forming part of a triangulation carried out near Kingston in 1881-82. Four independent measurements were made to ascertain its length, and the results were:

1	1060.1 yards
2	1060.9
3	1060.6
4	1060.4
	4)4242.0
Mean=	1060.5

Here the squares of the residuals, *in tenths of yards*, are

1.	16
2.	16
3.	1
4.	1
Total	34

And the probable error of the mean is

$$\frac{\sqrt{34}}{4} \times 0.674489 \times 3{\cdot}6 \text{ inches.}$$

$$= 3{\cdot}546 \text{ inches.}$$

PART II.

GEODESY.

CHAPTER I.

THE FIGURE OF THE EARTH.

Geodesy is a word of Greek derivation, and signifies "division of the earth." Broadly speaking, it comprises all surveying operations of such magnitude that the figure of the earth has to be taken into consideration.

The earth is an oblate spheroid—that is, the figure formed by the revolution of an ellipse round its minor axis—the polar axis being shorter than the equatorial by about 26·88 miles. This has been proved in two ways. Firstly, by pendulum experiments, which show that the force of gravity increases from the equator towards the poles; secondly, by actual measure-

ments of portions of meridianal arcs. A little consideration will show that if the curvature of a meridianal arc is elliptical, and therefore decreasing towards the poles, the length on the earth's surface of a degree of latitude must be greater in high than in low latitudes. That is, if A and B are two points on a meridian near the equator, but differing by a certain amount in astronomical latitude, and C and D two points on a meridian in a high latitude and also differing by the same amount, then if the distances A B and C D are measured on the ground, A B will be found to be less than C D. This has actually been done at various parts of the earth's surface—Lapland, Peru, France, Russia, (where an arc of over 25° was measured,) India, Algiers, South Africa, &c. The method adopted is to measure a base very accurately, and from it to connect by means of a chain of triangulation two distant stations which are as nearly as possible on the same meridian. This being done we can calculate the actual distance from one of the stations to the point where a perpendicular drawn to the meridian of that station from the other station meets the meridian. The latitudes of the two stations are found by very careful astronomical observations, and their difference, taken in connection with the calculated distance on the meridian, gives the curvature of the arc, since the radius of curvature is the measured distance divided by the difference of latitude in circular measure. There is, however, one source of error in determinations of this kind. In finding the latitudes of stations we are in general dependent on the direction of the plumb line; and should there, as often happens, be a local abnormal deviation of the latter from the true perpendicular, the resulting latitude will be erroneous. This was proved many years ago by taking the latitudes of two stations on opposite sides of a mountain in Perthshire, and measuring the true horizontal distance between them, when it was found that the

difference of the latitudes as given by the astronomical observations was by no means the true one.

The experiment was repeated in 1855 at Arthur's seat near Edinburgh. Three stations, A,B,and C, were taken; B on the summit of the hill, and the other two at opposite sides of it. The astronomical differences of latitude of the stations, as obtained by a long series of observations, were as follows:

Betweeen A and B.........25″.53
" B and C.........17″

while the actual differences, as found by triangulation, were:

Between A and B.........24″,27
" B and C.........14″.19

As a rule, the deviation seldom exceeds a few seconds except in the neighbourhood of great mountain masses, as at the foot of the Himalayas, where it is as much as 30″.

Where there is considerable deviation in level countries it is no doubt caused by neighbouring portions of the earth's crust being either denser or lighter than the average. The actual amount of deviation at a station may be ascertained as follows. A series of other stations symmetrically grouped round it are chosen, and the latitude and longitude of each obtained by astronomical observations. The actual distance and azimuth of the central station from each of the others being known by triangulation, its latitude and longitude may be calculated from that of each of them separately, and the mean of the results may be taken as nearly the truth, since the errors caused by plumb line deviations at the various stations will probably counteract each other. This mean latitude and longitude of the central station being compared with the latitude and longitude as obtained by

astronomical observations will give the deviation of the plumb line.

If a and b are the semi-major and semi-minor axes of an ellipse, the distance of the centre from either focus is $\sqrt{a^2-b^2}$, and this quantity divided by a is called the "eccentricity." This is generally written e. The quantity $\frac{a-b}{a}$ is called the "compression" or "ellipticity," and is denoted by c. The latest calculations make the compression of the earth about $\frac{1}{293}$, the ratio of the semi-axes being believed to be 292 to 293. The true measure of the compression is the difference of the semi-axes divided by the mean radius of curvature of the spheroid. The equator has also been found to be elliptical, its major axis being about 500 yards longer than its minor axis.

It should be noted that the expression e has different meanings in different books. English writers occasionally employ it for the *compression* or *ellipticity*, while in American books it is used in the same sense as here, namely, for the *eccentricity*. Even in different chapters of the same work the letter e is often used both for the compression and the eccentricity.

The accompanying figure represents a section of the earth. PP′ is the polar axis, QE an equatorial diameter, C the centre, F a focus of the ellipse, A a point on the surface, A T a tangent at A, and Z A O perpendicular

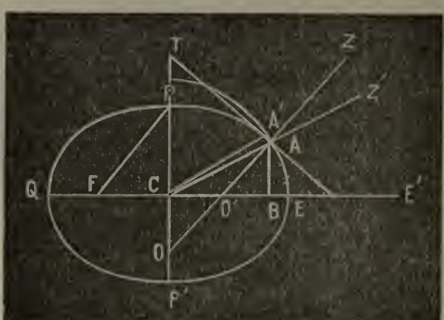


Fig. 30.

to A T. Z' is the geocentric zenith, and Z' C E′ is its declination. The latter is called the *geocentric or reduced lati-*

tude of A. $Z\,O'\,E'$ is the *geographical or astronomical* latitude, $Z\,A\,Z'$ or C A O is called the *reduction of the latitude.* It is evident that the geocentric is always less than the geographical latitude.

Let C E$=a$. C P$=b$. Let c be the compression and e the eccentricity.

$$c=\frac{a-b}{a}=1-\frac{b}{a}$$

$$e=\frac{CF}{CE}=\frac{CF}{PF}$$

$$\frac{CF^2}{CE^2}=\frac{PF^2-PC^2}{CE^2}=1-\frac{PC^2}{CE^2}$$

That is, $e^2=1-\dfrac{b^2}{a^2}=1-(1-c)^2$

or, $e=\sqrt{2c-c^2}$ (1)

TO FIND THE REDUCTION OF THE LATITUDE.

Taking the centre of the ellipse as the origin of axes, the equation of the ellipse will be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Let φ be the geographical latitude
φ' " geocentric "

We have, $\tan\varphi=-\dfrac{d\,x}{d\,y}$

and from the triangle ACB, $\tan\varphi'=\dfrac{y}{x}$

Differentiating the equation of the ellipse, we have

$$\frac{y}{x}=-\frac{b^2}{a^2}\frac{dx}{dy}$$

or,

$$\tan\varphi'=\frac{b^2}{a^2}\tan\varphi=(1-e^2)\tan\varphi \qquad (2)$$

To find the reduction, or $\varphi-\varphi'$, we use the general development in series of an equation of the form

$\tan x=p\tan y$, which is
$x-y=q\sin 2y+\frac{1}{2}q^2\sin 4v+$&c.

$$\text{in which } q = \frac{p-1}{p+1}$$

Applying this to the development of (2) we find, after dividing by sin 1″ to reduce the terms of the series to seconds, and putting $x=\varphi'$, $y=\varphi$.

$$\varphi - \varphi' = -\frac{q}{\sin 1''}\sin 2\varphi - \frac{q^2}{2\sin 1''}\sin 4\varphi - \&c. \quad (3)$$

$$\text{in which } q = \frac{p-1}{p+1} = \frac{1-e^2-1}{1-e^2+1} = -\frac{e^2}{2-e^2}$$

The known value of e gives q, and thence $\varphi - \varphi'$ for any given value of φ.

N.B.—q is negative, and q^2 is very small compared with it ; therefore $\varphi - \varphi'$ is positive.

In some books on geodesy the expression "reduc~~tion~~ed ~~of~~ ~~the~~ latitude" is applied to the angle A′ C E, where A′ is the point in which B A produced meets the circle described with centre C and radius C E. Let this angle be φ''.

$$\text{Then } \frac{\tan\varphi'}{\tan\varphi''} = \frac{\text{B A}}{\text{B A}'} = \frac{b}{a} \qquad \frac{b^2}{a^2}$$

by the properties of the ellipse. And since $\tan\varphi' = \frac{a^2}{b^2}\tan\varphi$

$$\text{we have} \quad \frac{\tan\varphi}{\tan\varphi''} = \frac{a^2}{b^2}\tan\varphi' \div \frac{a}{b}\tan\varphi' = \frac{a}{b}$$

$$\text{and} \quad \frac{\tan\varphi}{\tan\varphi''} = \frac{\tan\varphi''}{\tan\varphi'}$$

TO FIND THE RADIUS OF THE TERRESTRIAL SPHEROID FOR A GIVEN LATITUDE.

Let ρ (or A C) be the radius for latitude φ.

We have, $\rho = \sqrt{x^2 + y^2}$

To express x and y in terms of φ, we have, substituting $1-e^2$ for $\frac{b^2}{a^2}$ in the equation to the ellipse and its differential equation,

$$x^2+\frac{y^2}{1-e^2}=a^2$$

$$\frac{y}{x}=(1-e^2)\tan\varphi$$

whence, by elimination, we find

$$x=\frac{a\cos\varphi}{\sqrt{1-e^2\sin^2\varphi}}$$

$$y=\frac{(1-e^2)\,a\sin\varphi}{\sqrt{1-e^2\sin^2\varphi}}$$

and hence, $\rho=a\left(\frac{1-2\,e^2\sin^2\varphi+e^4\sin^2\varphi}{1-e^2\sin^2\varphi}\right)^{\frac{1}{2}}$ (4)

TO FIND THE LENGTH OF THE GREAT NORMAL, A O, FOR A GIVEN LATITUDE.

From the figure we have

$$\text{Great normal}=\frac{\rho\cos\varphi'}{\cos\varphi}$$

$$=\frac{a}{\sqrt{1-e^2\sin^2\varphi}}=\mathcal{N} \tag{5}$$

In future the great normal will be spoken of simply as the "normal."

NOTE.—The length of a second of longitude in latitude φ will be $x\sin 1''$, or

$$\frac{a\cos\varphi\sin 1''}{\sqrt{1-e^2\sin^2\varphi}}=\mathcal{N}\cos\varphi\sin 1''$$

TO FIND THE RADIUS OF CURVATURE OF THE TERRESTRIAL MERIDIAN FOR A GIVEN LATITUDE.

Denote this radius by R.

We have, from the Differential Calculus,

$$R=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

From the equation to the ellipse we have

$$\frac{dy}{dx} = -\frac{b^2}{a^2}\,\frac{x}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{b^4}{a^2 y^3}$$

whence $$R = \frac{(a^4 y^2 + b^4 x^2)^{\frac{3}{2}}}{a^4 b^4}$$

Observing that $b^2 = a^2(1-e^2)$, we find, by substituting the values of x and y in terms of φ (page 115.)

$$R = \frac{a(1-e^2)}{(1-e^2 \sin^2 \varphi)^{\frac{3}{2}}} = \mathcal{N}^3 \frac{1-e^2}{a^2} \qquad (6)$$

This last equation gives the length of a second of latitude at a given latitude, since it is equal to R sin 1″

The following formula is sometimes used for the radius of curvature of the meridian,

$$R = \frac{a+b}{2} - \tfrac{3}{2}(a-b)\cos 2\varphi$$

It also admits of proof that the normal at any point is the radius of curvature of a section of the earth's surface through the normal and at right angles to the meridian.

From equations (5) and (6) we see that the normal at any point is always greater than the radius of curvature of the meridian at that point.

If the earth were a sphere the shortest line on its surface between any two points A and B (otherwise called the *geodesic line*) would be an arc of a great circle, and the azimuth of A at B would differ from that of B at A by 180°. + Convergence of meridians A & B. But on the surface of a spheroid the geodesic line is, except when both points are on the equator or on the same meridian, a curve of double curvature. The two azimuths, also, will not, except in certain cases, differ from each other by exactly ~~180°~~ by the above amount. The reason of this is that the vertical plane at A passing through B will not

coincide with the vertical plane at B passing through A. These two planes will, of course, intersect at A and B, but their intersections with the surface of the spheroid will be different curves, and will enclose between them a space. In addition to these two lines and the geodesic line there will also be what is known as the *line of alignment* of the two points—that is the line on every point of which the line of sight of the telescope of a theodolite in perfect adjustment and truly levelled would, when directed on one station, intersect the other on the telescope being turned over.

CHAPTER II.

GEODETICAL OPERATIONS.

The methods adopted in the old world for mapping large tracts of country have been reversed in America. Instead of starting from carefully measured bases, and carrying out chains of triangulation connecting various principal points in such a manner that the relative positions of the latter with respect to each other may be ascertained within a few inches, though several hundred miles apart, the system pursued (if we except the U. S. Coast Survey and some other triangulations) has been to take certain meridians and parallels of latitude intersecting each other; to trace and mark out these meridians and parallels on the ground; to divide the figures enclosed by them into blocks or "checks;" and to further subdivide the latter into townships, sections, and quarter sections. Although the method of triangulation is incomparably the most accurate, the American plan has the advantage of rapidity and cheapness. As the latter is very simple, and is fully explained in the Canadian Government Manual of Survey, it will not be further touched upon here.

At the commencement of a triangulation a piece of tolerably level ground having been selected, a base line,

generally from 5 to 10 miles long, is measured with the very greatest care, and from it either a network or a chain of triangles is started. In the former case the triangles are expanded as rapidly as possible till they are large enough to cover the whole country with a network of primary triangles. This is done by taking angles from the extremities of the base to certain selected points or "signals," (often on mountain tops), and calculating their distances by trigonmetry. The instrument is then placed at each of these new stations, and angles taken from them to still more distant points, the calculated lines being used as new base lines. This process is repeated and extended till the whole district is covered by these primary triangles, the sides of which should be as large as possible.

Smaller, or *secondary*, triangles are formed within the primary ones to fix the positions of important points which may serve as starting points for traverses, &c. Tertiary triangles are sometimes formed within the secondary ones.

The size of the primary triangles varies according to circumstances. Their sides are often from 30 to 60, or even 100 miles, and in one instance as much as 170 miles. The longest side in the British triangulation was 111. The sides of the secondary triangles are from about 5 to 20 miles, and those of the tertiary triangles five or less.

The larger triangles should be as nearly equilateral as circumstances admit of. The reason for having them so is that with this form small errors in the measurement of their angles will have a minimum effect on the calculated lengths of the sides. Such triangles are called "well-conditioned" ones.

The original base has to be reduced to the level of the sea—that is, the true distance between the points where verticals through its ends intersect the sea level must be

ascertained. The exact geographical position of one end, and the azimuth of the other with respect to it, must of course be known. The angles of all the principal triangles must be measured with the greatest exactness that the best instruments admit of, the lengths of the sides calculated by trigonometry, and their azimuths worked out. The work (when carried on on a very large scale) is still further complicated by the earth's surface being not a sphere but a spheroid. The accuracy of the triangulation is tested by what is called a "base of verification." That is, a side of one of the small triangles is made to lie on suitable ground, where it can be actually measured. Its length, as thus obtained, compared with that given by calculation through the chain of triangles, shows what reliance can be placed on the intermediate work.

As instances: The triangulation commenced at the Lough Foyle base in the North of Ireland was carried through a long chain of triangles to a base of verification on Salisbury plain, and the actual measured length of the latter was found to differ only 5 inches from the length as calculated through several hundred miles of triangulation. An original base was measured at Fire Island, near New York, and afterwards connected with a base of verification on Kent Island in Chesapeake Bay. The actual distance between them was 208 miles, and the distance through the 32 intervening triangles 320. The difference between the computed and measured lengths of the base of verification was only 4 inches. In Algiers, two bases about 10 kilometres long were connected by a chain of 88 triangles. Their calculated and measured distances agreed within 16 inches.

If the country to be triangulated is very extensive—as, for instance, in the case of India—instead of covering it with a network of triangulation, it may be intersected in the first place by chains of triangles, either single or double,

and bases measured at certain places;usually where these chains meet. In India the chains run generally either north and south or east and west, and form a great frame or lattice work on which to found the further survey of the country. A double chain of triangles forms, of course, a series of quadrilateral figures, in each of which both the diagonals, as well as the sides, may be calculated.

The following is a brief account of the measurements of some celebrated base lines :

In 1736 a base line had to be measured in Lapland for the purpose of finding the length of an arc of the meridian by triangulation. A distance of about 9 miles was measured in mid winter on the frozen surface of the River Tornea. By means of a standard *toise* brought from France, a length of exactly 5 toises (about 32 feet) was marked on the inside wall of a hut, and eight rods of pine, terminated with metal studs for contact, cut to this exact length. It had been previously ascertained that changes of temperature had no apparent effect on their length. The surveying party was divided into two, each taking four rods, and two independent measurements of the base were made, the results agreeing within four inches. The time occupied was seven days. The rods were probably placed end to end on the surface of the snow.

The same year a base 7.6 miles long was measured near Quito in Peru, at an altitude of nearly 8000 feet. The work occupied 29 days. Rods 20 feet long, terminated at each end by copper plates for contact, were used. The rods were laid horizontally, changes of level being effected by a plummet suspended by a fine hair. The rods were compared daily with a toise marked on an iron bar which had been laid off from a standard toise brought from Paris. This base was the commencement of a chain of triangles for the measurement of a meridianal arc. Three years later another base, 6.4 miles long,

was measured near the south end of this chain, and only occupied ten days. The party was divided into two companies which measured the line in opposite directions.

The trigonometrical survey of Great Britain was commenced by the measurement of a base on Hounslow Heath, which was chosen from the great evenness and openness of the ground. Three deal rods, tipped with bell metal and 20 feet long, were used at first. But it was found that they were so affected by changes in the humidity of the atmosphere that glass tubes of the same length, of which the expansion for temperature had been ascertained, were substituted, the temperatures of the tubes being obtained by attached thermometers. The length of the base when reduced to the sea level and 62° Faht. was 9,134$\frac{2}{3}$ yards. This distance was subsequently re-measured with a steel chain 100 feet long, consisting of 40 links half an inch square in section. A second similar chain was used as a standard of comparison. The chain was laid in five deal coffers carried on trestles, and was kept stretched by a weight of 28 pounds. The exact end of each chain's length was marked by a slider on the top of a post. The two measurements (glass tubes and steel chains) agreed within two inches.

Two bases, each about 7$\frac{1}{4}$ miles long, were subsequently measured in France—one near Paris, the other at Carcassonne in the south. Four rods were used. They were composed of two strips of metal in contact (platinum and copper), forming a metallic thermometer, carried on a stout beam of wood. Each rod was supported on two iron tripods fitted with levelling screws, and there was an arrangement for measuring their inclination.

The Lough Foyle base was measured with Colby's compensation bars; an arrangement in which the unequal expansions and contractions of two parallel bars of different metals (brass and iron), 10 feet long, are utilized to keep

two platinum points at an invariable distance from each other. The bars were arranged in line on roller supports in boxes laid on trestles, and the intervals between the bars were measured by similar short compensating bars six inches long, at each end of which was a microscope into the focus of which the platinum points of the measuring rods were bought by means of micrometer screws. The line commenced at a platinum dot let into a stone pillar, and the rods were kept in a true straight line by a transit theodolite. About 250 feet a day was measured on an average; 400 feet of the line was across a river, the boxes being laid on piles about 5 feet apart. Eight miles were measured thus, and two more were subsequently added to the base by triangulation. The Salisbury Plain base was measured in the same way. Colby's bars were subsequently used for ten bases in India, but were not found to give very reliable results there.

An improvement on Colby's arrangement is the compensating apparatus used in the United States coast survey. It consists of a bar of brass and a bar of iron, a little less than six metres long and parallel to each other. The bars are joined together at one end, but free to move at the other. Their cross-sections are so arranged that while they have equal absorbing surfaces their masses are inversely as th~~is~~eir specific heats, allowance being made for their difference of conducting power. The brass bar is the lowest, and is carried on rollers mounted in suspending stirups. The iron bar rests on small rollers fastened to it which run on the brass bar.

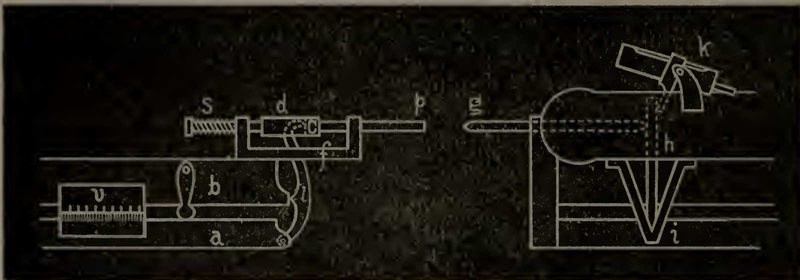


Fig. 31.

The annexed figure shows the arrangement at the two ends, the left hand part being the compensation end. It will be seen that the lever of compensation (*l*) is pivoted on the lower bar (*a*), a knife edge on its inner side abutting on the end of the iron bar (*b*.) This lever terminates at its upper end in a knife edge (**c**) in such a position that whatever be the expansion or contraction of the bars it always retains an invariable distance from their other end. This knife edge presses against a collar in the sliding rod (*d*), moving in a frame (*f*) fixed to the iron bar, and is kept back by the spiral spring (*s*). The rod is tipped with an agate plane (*p*) for contact. The vernier (*v*) serves to read off the difference of lengths of the bars as a check.

At the other end where the bars are united a sliding rod terminates in a blunt horizontal knife edge (*g*), its inner edge abutting against a contact lever (*h*) pivoted at (*i*). This lever, when pressed by the sliding rod, comes in contact with the short tail of the level (*k*), which is mounted on trunnions and not balanced. For a certain position of the sliding rod this bubble comes to the centre, and this position gives the true length of the measuring bar. Another use of the level is to ensure a constant pressure at the points of contact, *p* and *g*. To the lever and level is attached the arm of a sector which gives the inclination of the bar.

The bars are enclosed in a spar-shaped double tin tubular case, the air-chamber between the two cases preventing rapid changes of temperature. The ends are closed, the ends only of the sliding rods projecting. The level, sector, and vernier, are read through glass doors. The tubes are painted white and mounted on a pair of trestles. Two of these bars are used in measuring. They are aligned by a transit.

On one base, seven miles long, measured with this apparatus, the greatest supposable error was computed, from re-measurement, to be less than six-tenths of an inch. On another base, six and three quarter miles long, the probable error was less than one-tenth of an inch, and the greatest supposable error less than three-tenths.

This apparatus has been tested by measuring a base in Georgia three times, twice in winter and once in summer, at temperatures ranging from 18° to 107° Faht. The discrepancies of the three measures with their respective means were, in millimetres,—8.10,—0.32, and +8.41.

It has been found, however, that the apparatus is not quite perfect, its true length depending on whether the temperature is rising or falling.

The amount of accuracy to be aimed at in measuring a base depends on the extent of the survey. For small surveys it may be sufficient to measure the base two or three times with a steel tape which is kept compared with a standard. The tape should be stretched each time to a constant tension by means of a spring balance. It is a good plan to mark the end of each chain on a small piece of plank, which is made to adhere to the ground by means of pointed spikes on its under surface.

Pine rods, well seasoned, baked, boiled in drying oil, painted and varnished, may be used. They should either be levelled or have their angle of inclination read. If the

ground is uneven they may be levelled on trestles with sliding telescopic supports. The ends of the rods should be capped with metal, either wedge-shaped or hemispherical in form, and either placed in actual contact, or the spaces between them measured by graduated glass wedges. If the end of one rod has to be placed on a different level to that of the next a fine plumb line may be used; or the rods may have fine lines marked at each end of the unit of length, so that one rod may be made to overlap the other with the two marks exactly corresponding. This plan answers well on ice.

Before measuring an important base it is usual to make a preliminary approximate measurement of the line, and also to get an accurate section of it by levelling. Suitable points are selected for dividing it into sections, and these points are accurately adjusted into line by means of a transit at one end. It may happen, however, that it is impracticable to have all the segments in a straight line, in which case the angles they make with each other must, of course, be exactly measured. Any deviation also from a true horizontal line must be recorded in order that the base may be reduced to the sea level. The ends of the base, as well as of the sections, are generally marked by microscopic dots on metallic plates let into massive stones embedded in masonry, and are thus permanently recorded. The mark itself may be a minute cross on a piece of brass, or a dot on the end of a platinum wire set vertically in a piece of lead run into a hole in the stone.

If the rods used in measuring the base expand and contract with changes of temperature the latter must be recorded at regular intervals of time, as the rods are at their true length only when at a certain standard temperature.

If the base, or any portion of it, is not level, its inclina-

tion must be measured for the purpose of reducing it to its horizontal projection.

Let B be the length of an inclined portion, b the length reduced to the horizontal, and θ the angle of inclination. Then $b = B \cos \theta$.

As θ is generally a very small angle and need not be known exactly, it is better to compute the excess of B above b. If θ is given in minutes we have

$$B - b = B\ (1 - \cos\ \theta) = 2\ B\ \sin^2 \frac{\theta}{2}$$

$$= \tfrac{1}{2}\ B\ \theta^2\ \sin^2\ 1'$$

$$= 0.0000004231\ \theta^2\ B.$$

If the base is intersected by a ravine or creek which cannot be conveniently measured across we may proceed as follows:

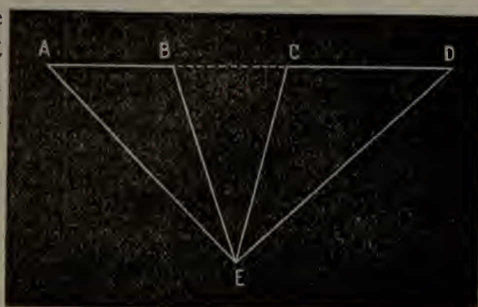


Fig. 32.

Let AB CD be the base, and BC the interrupted portion (Fig. 32). Let AB $= a$, CD $= b$, and BC $= x$. Take an exterior station E and measure the angles AEB (α) AEC (β) and AED (γ). Then if φ is such an angle that

$$\tan^2 \varphi = \frac{4\,a\,b}{(a-b)^2} \times \frac{\sin \beta \sin (\gamma - \alpha)}{\sin \alpha \sin (\gamma - \beta)}$$

It may be proved that

$$x = -\frac{a+b}{2} \pm \frac{a-b}{2\ \cos \varphi}$$

The base is, of course, $a + b + x$.

If the nature of the ground necessitates an angle C between two portions of the base A C, C B, we can find the direct distance A B thus: The angle C (which is very obtuse) is measured with great care. Let $180°-C=\theta$, $A\,C=b$, $C\,B=a$, and $A\,B=c$.

Then $c^2=a^2+b^2+2\,ab\cos\theta$

and (if θ is not more than $10°$)

$$\cos\theta=1-\frac{\theta^2}{2},\ \text{nearly.}$$

$$\therefore c^2=a^2+b^2+2\,a\,b-a\,b\,\theta^2$$
$$=(a+b)^2-a\,b\,\theta^2$$
$$(a+b)^2\left(1-\frac{a\,b\,\theta^2}{(a+b)^2}\right)$$

$$\text{and } c=(a+b)\left\{1-\frac{a\,b\,\theta^2}{(a+b)^2}\right\}^{\frac{1}{2}}$$

$$=(a+b)\left\{1-\tfrac{1}{2}\frac{a\,b\,\theta^2}{(a+b)^2}+,\ \&c.,\right\}$$

$$=a+b-\frac{a\,b\,\theta^2\sin^2 1'}{2\,(a+b)}$$

$$=a+b-0.0000000423\,1+\frac{a\,b\,\theta^2}{a+b}$$

θ being in minutes.

To reduce a measured base to the sea level we must know the height of every portion of it in order to get its mean height. Let l be the length of a rod, and h its height; l' its projection on the sea level, and r the radius of the earth.

$$\text{Then}\qquad \frac{l'}{r}=\frac{l}{r+h},$$

$$\text{or}\qquad l'=\frac{l\,r}{r+h}=l\left\{1-\frac{h}{r}\right\},\ \text{nearly.}$$

If n be the number of rods in the base and $n\,l=L$; then the length of the base reduced to the sea level will be $L\left\{1-\frac{1}{r}\frac{\Sigma\,(h)}{n}\right\}$; $\frac{\Sigma\,(h)}{n}$ being the mean height of all the rods.

The base thus reduced is a curve. To find the length of its chord we should have to subtract a very minute quantity, namely, the cube of the base divided by 24 times the ~~mean~~ square of the earth's radius.

MEASUREMENT OF BASES BY SOUND.

This is a rough method which has sometimes to be adopted in hydrographic surveys of extensive shoals which have no points above water. It should, if possible, only be adopted in calm dry weather. The velocity of sound in air is 1089.42 feet per second at 32° Faht. It is almost uneffected by the wind, the baromet~~er~~ric pressure, and the hygrometric condition of the air. The observers are posted at both ends of the base and are provided with guns, watches, and thermometers. When the gun at one end is fired the observer at the other notes the interval in seconds and fractions between the flash and the report. The guns are fired alternately from both ends at least three times, a preparatory signal being given.

The value of the velocity of sound given above must be corrected for the temperature (t°) by multiplying it by the quantity

$$\sqrt{1+(t^\circ-32^\circ)\times 0.00208.}$$

Of course the distance is the corrected velocity multiplied by the mean of the observed intervals of time. The errors of observation are always considerable, but are no greater for long distances than for short ones.

ASTRONOMICAL BASE LINES.

In cases where no suitable ground for a measured base is available two convenient stations may be selected as the ends of an imaginary base line, and their latitude and longitude, with the azimuth of one from the other, ascertained by astronomical observations. We shall then have the length and position of the base with more or less accuracy, and a triangulation can be carried on

from it. The base chosen should be as long as possible, but not greater than one degree. None of the sides of the triangles should be greater than the base. The azimuths of the sides being known, the positions of the observed points can be plotted by co-ordinates.

If the zenith telescope and portable transit telescope are used the latitude can be determined within 10″, the longitude and azimuth within 30″. With the sextant these errors are at least doubled. Differences of longitude may be determined by flashing signals.

CHAPTER III.

TRIANGULATION.

Having discussed the measurement of base lines we have now to consider the triangulation. It is evident that the latter may be commenced without waiting to complete the former. The first thing to be done is to select the stations and to erect the necessary points to be observed, or "signals" as they are called. In a hilly country the mountain tops naturally offer the best stations, as being conspicuous objects and affording the most distant views. In this case the size of the triangles is only limited by the distance at which the signals can be observed. Thus, in the Ordnance Survey of Ireland the average length of the sides of the primary triangles was 60 miles, while some were as much as 100. In the triangulation which was carried in 1879 across the Mediterranean between Spain and Africa, by means of the electric light, signals were observed at a distance of 170 miles.

In a flat country lofty signals have to be erected, not only that they may be mutually visible, but in order that the rays of light may not pass too close to the surface of the earth, as they would be thereby too much affected by refraction. In the U. S. Coast Survey six feet is considered the limit advisable. If h', h'', are the heights of two

signals in feet and d their distance in miles, then, on a flat country or over water, they will not, under ordinary circumstances, be visible to each other if d is greater than $\frac{4}{3}(\sqrt{h'} + \sqrt{h''})$ The most difficult country of all in which to carry out a triangulation is one that is flat and covered with forest.

Formerly, conspicuous objects, such as the points of church spires, were commonly used as signals; but of late years this has not been done, because in all large triangles it is necessary to measure all the three angles, and this cannot well be done directly in the case of such objects. The form of the signals varies much. Whatever kind be used the centre of the theodolite must be placed exactly under or over the centre of the station, and if a scaffolding has to be employed the portion on which the instrument is supported must be disconnected with that on which the observers stand. One kind of signal is a vertical pole with tripod supports, the pole being set up with its summit exactly over the station. It may be surmounted by two circular disks of iron at right angles to each other. A piece of square boarding, painted white with a vertical black stripe about four inches wide, can be seen a long way off. Flags may be used, but are not always easy to see. A good form of signal is a hemisphere of silvered copper with its axis vertical. This will reflect the rays of the sun in whatever position the latter may be, but a correction for "phase" will be required, as the rays will be reflected from different parts of the hemisphere according to the time of day. The ordinary signal used in the United States is a pole 10 to 25 feet high, surmounted by a flag, and steadied by braces. With respect to its diameter, the rule is that for triangles with sides not exceeding five miles it should not be more than five inches. If more than five miles, five to eight inches. Various other forms of special signals are used in the U. S. Coast Survey. Amongst others may be mentioned a

pyramid of four poles, with its upper portion boarded over and terminating in a point, directly under which the theodolite is placed. In England double scaffoldings as high as 80 feet were used, the inner scaffolding carrying the instrument and the outer one the observers. In Russia a triangulation had to be carried on over an arc of more than 500 miles across a flat swampy country covered with impenetrable forests, and scaffoldings of as much as 146 feet high had to be erected. On the prairies of the Western States towers have had to be built; as also has been done in India, where solid towers were used at first, but were afterwards superseded by hollow ones, which allowed the instruments to be centred vertically over the stations. The centres of trigonometrical stations are generally indicated by a well-defined mark on the upper surface of a block of stone buried at a sufficient distance below the surface. In the Algerian triangulation the stations were marked by flat-topped cones of masonry having a vertical axial aperture communicating with the station mark.

In sunlight, stations may be rendered visible at a great distance by means of the heliostat, and at night the electric light is now much employed. In the triangulation across the Mediterranean already alluded to the signal lights were produced by steam-engines of six-horse power working magneto-electric machines. These lights were placed in the focus of a reflector 20 inches in diameter, consisting of a concavo-convex lens of glass with the convex surface silvered. The curvatures of the surfaces corrected the lens for spherical aberration, and it threw out a cone of white light, having an amplitude of 24′, which was directed on the distant station by a telescope. A refracting lens, eight inches in diameter, was also used, and threw the light one hundred and forty miles. There were two Spanish stations fifty miles apart. Mulhacen, 11,420 feet high, and Tetica, 6,820 feet. The

two Algerian stations, 3,730 and 1,920 feet, were 66 miles apart, and were each distant from Mulhacen about 170 miles. The labour of transporting the necessary machinery, wood, water, &c., to such a height as Mulhacen was very great. It was twenty days after everything had been got ready before the first signal light was made out across the sea. After that the observations were carried on uninterruptedly. In France, night observations have been carried on by means of a petroleum lamp placed in the focus of a refracting lens of eight inches diameter.

MEASURING THE ANGLES.

Of late years the only instruments used for measuring the angles of a triangulation have been theodolites of various sizes; the larger natures being really "alt-azimuth" instruments. The more important and extended the survey the larger and more delicate are the instruments employed. In the great triangulation of India theodolites of 18 and 36 inches diameter were used, the average length of the triangle sides being about 30 miles. For the Spanish-Algierian triangulation they had theodolites of 16 inches diameter read by four micrometers. In the United States Coast Survey the large theodolites have diameters of 24 and 30 inches. For the secondary and tertiary triangles smaller instruments are used. The method of taking the angles varies with the nature of the instrument. The smaller ones have usually two verniers. Those of about 8 inches diameter have three, while the arcs of the larger ones are read by micrometers, of which some have as many as five. In all cases errors due to unequal graduation and false centreing are almost entirely eliminated by the practice of reading all the verniers or micrometers, and taking the same angles from different parts of the arc. It is usual to measure all important angles a large number of times.

Of the smaller theodolites there are two kinds, the repeating and the reiterating. In a repeating theodolite the lower horizontal plate is free to revolve if necessary, and thus, when the two plates are clamped at any particular reading, the telescope can be directed to any point required. In a reiterating theodolite the lower plate is fixed to the stand, and when the instrument is set up for the purpose of measuring a horizontal angle it is quite a chance what point of the graduation the angle will have to be measured from. This form is not so convenient as the other for general purposes, but it has the advantage of superior stability, and is therefore preferred for running straight lines. The 6-inch reiterating transit theodolites used on the Canadian Government surveys have three verniers. If an angle is read off on each, and the telescope then turned over and the angle remeasured, we shall have six measures from six equidistant parts of the arc, the mean of which is taken. In all cases, after reading an angle or a round of angles, the telescope should be again directed on the first object observed, so as to make sure that the reading is the same and that nothing has slipped.

The method of "repeating" an angle is this. When the telescope has been directed on the second, or right hand, object, and clamped, instead of reading the vernier, the lower plate is set free, and the two together revolved till the telescope is again set on the left hand object. The lower plate is then clamped, the upper one set free, and the telescope directed on the right hand object. The vernier may now be read, when the angle will be half the reading—or the repeating process may be continued for any number of times, and the whole arc passed over, divided by the number of repetitions. The object of this process is to eliminate errors of observation and graduation. Owing, probably, to slipping of the plates, it does not usually give such good results as might be expected.

TO REDUCE A MEASURED ANGLE TO THE CENTRE OF A STATION.

It may happen than an inaccessible object—such as the summit of a church spire—has to be used as an angle of an important triangle. It cannot, of course, be measured directly, but it may be found indirectly as follows:

Let ABC be the triangle and A the inaccessible point. Take a contiguous point A′ and measure the angles ABC, BCA, BA′C, AA′B. Calculate or otherwise obtain the distance AA′ on plan.

Call BAC, A; BA′C, A′; ABA′, α; and ACA′, β.

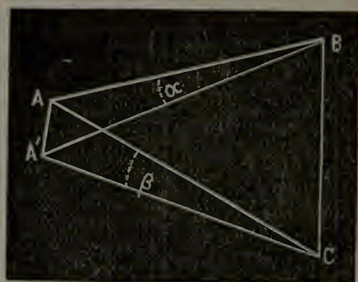


Fig. 33.

Now $A+\alpha=A'+\beta$.

Therefore $A=A'+\beta-\alpha$.

Also, AB and AC are known, and

$$\begin{cases} AB \sin\alpha = AA' \sin AA'B \\ AC \sin\beta = AA' \sin AA'C \end{cases}$$

or, since α and β are very small angles, if they are taken in seconds,

$$\begin{cases} AB \times \alpha \sin 1'' = AA' \sin AA'B \\ AC \times \beta \sin 1'' = AA' \sin AA'C \end{cases}$$

$$\text{Therefore, } A=A'-\frac{AA' \sin AA'B}{AB \sin 1''}+\frac{AA' \sin AA'C}{AC \sin 1''}$$

CORRECTION FOR PHASE OF SIGNAL.

If the sun shines on a reflecting signal—such as a polished cone, cylinder, or sphere—the point observed will, in general, be on one side of the true signal, and a correction will have to be made in the measured angle. The following is the rule in the case of a cylinder.

Let r be the radius of the base of the cylinder, Z the horizontal angle at the point of observation between the sun and the signal, and D the distance.

Then, the correction $= \pm \dfrac{r \cos \frac{Z}{2}}{D \sin 1''}$

The proof is very simple.

In the case of a hemisphere the value of r will depend on the sun's altitude. If we call the latter A, r will become $r \cos \dfrac{A}{2}$, which must be substituted for r in the above equation.

TO REDUCE AN INCLINED ANGLE TO THE HORIZONTAL PLANE.

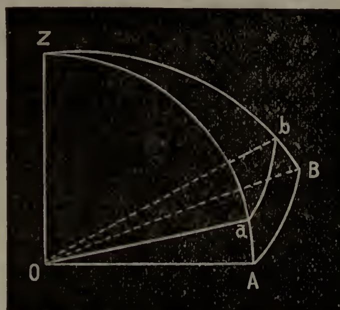


Fig. 34.

It often happens, as in the case of angles measured with the sextant or repeating circle, that the observed angle is inclined to the horizontal, and a reduction is necessary to get the true horizontal angle. In Fig. 34, let O be the observer's position, a and b the objects, and a O b the observed angle. If Z is the zenith, and vertical arcs are drawn through a and b, meeting the horizon in A and B, then A ~~B O~~ O B is the angle required. $a\,Z\,b$ is a spherical triangle, and by measuring the vertical angles Aa, Bb, we shall have its three sides, since Z A and Z B are each 90°. Also, $a\,Z\,b$=A O B. If we call ab, h; Za, z; and Zb, z', we can obtain a Z b from the equation

$$\sin \frac{a\,Z\,b}{2} = \left\{ \frac{\sin\,(s-z)\,\sin\,(s-z')}{\sin z \sin z'} \right\}^{\frac{1}{2}}$$

$$\text{where } s = \frac{h+z+z'}{2}$$

The arcs Aa, Bb are generally small, and the differ-

ence of z and z' therefore also small. The arcs may therefore be substituted for the sines, and we have for the correction (in seconds)

$$\text{AOB}-h=\left\{90^\circ-\frac{z+z'}{2}\right\}^2\tan\frac{h}{2}\sin 1''-\left\{\frac{z-z'}{2}\right\}^2\cot\frac{h}{2}\sin 1''$$

This formula is applicable when z and z' are within 3° of 90°.

If one of the objects is on the horizon we shall have

$$\text{AOB}-h=-2\left\{45^\circ-\frac{z}{2}\right\}^2\cot h\sin 1''$$

If, in addition, the angle h is 90° the correction will be nil.

THE SPHERICAL EXCESS.

The angles of a triangle measured by the theodolite are those of a spherical triangle; the reason being that at each station the horizontal plate when levelled is tangential to the earth's surface at that point. We must therefore expect to find that the three angles of a large triangle, when added together, amount to more than 180°; and this is actually the case. The difference is called the "spherical excess." From spherical trigonometry we know that its amount is directly proportional to the area of the triangle. In small triangles it is inappreciable. An equilateral triangle of 13 miles a side would have an excess of only one second. For one of 102 miles it would be one minute.

Taking for granted that the spherical excesses of two triangles are as their areas we can easily find the excess for a triangle of area s—thus: A trirectangular triangle has a surface of one-eighth that of the sphere, or $\frac{\pi r^2}{2}$, and its excess is 90°, or 324000″. The excess, in seconds, will therefore be equal to $\frac{2\times 324000}{\pi r^2}\times s$; r and s being, of course, in the same unit of measure.

Since s is very small compared with r^2 it may be obtained with sufficient accuracy for the purpose by treating the triangle as it were a plane one. We may thus use either of the formulæ

$$s = \frac{a\,b \sin C}{2}$$

$$\text{or} \qquad s = \frac{a^2 \sin B \sin C}{2 \sin (B+C)}$$

according to the data given.

Approximately, the spherical excess (in seconds) is the area in square miles divided by 75.5.

The expression for the spherical excess may be put in the form $\dfrac{a\,b \sin C}{2} \times \dfrac{1}{r^2 \sin 1''}$

For very large triangles we must take the eccentricity of the earth into account. The expression will then become

$$\frac{a\,b \sin C\,(1+e^2 \cos^2 L)}{2\,r^2 \sin 1''}$$

where e is the eccentricity of the earth, r the equatorial radius, and L the mean latitude of the three stations.

CORRECTING THE ANGLES OF A TRIANGLE.

In practice the sum of the three measured angles of a triangle is never what it ought to be, and they have to be corrected. Supposing that all three angles have been measured with equal care, the plan adopted is to add to or subtract from each of the angles one-third of the excess or deficiency. Thus, if E″ was the calculated spherical excess the three angles ought to amount to $180°+E''$. Supposing that they amounted to $180°+n''$ and that n were greater than E. Then we should subtract from each angle $\dfrac{n-E}{3}$

If some angles have been measured oftener, or with greater care, than others, the amount of correction to be

applied to each will be inversely as the *weights* attached to the results of the measurements.

In the Spain-Algiers quadrilateral triangulation the spherical excesses of the four triangles were

$$43''.50\,;\ 60''.7\,;\ 70''.73\,;\ 54''.16$$

and the errors of the sums of the observed angles were

$$+0''.18 \qquad -0''.54 \qquad +1''.84 \qquad +1''.12$$

CALCULATING THE SIDES OF THE TRIANGLES.

The next step is the calculation of the sides of the triangles. Treating the latter as spherical this may be done in three ways.

1. Using the ordinary formulas of spherical trigonometry. This is a very laborious method, and others which are simpler give equally good results.

2. Delambre's method. This consists in taking the chords of the sides, calculating the angles they make with each other, and solving the plane triangle thus found.

To reduce an arc a to its chord we have

$$\text{Chord}=2 \sin \tfrac{1}{2}\, a$$

or, if the arc be in terms of the radius,

$$\text{Chord}=a-\tfrac{1}{24}\, a^3 \quad *$$

The angles made by the chords are obtained by a well-known problem in spherical trigonometry.

3rd method, by Legendre's Theorem ; which is, that in any spherical triangle, the sides of which are very small compared to the radius of the sphere, if each of the angles be diminished by one-third of the spherical excess, the sines of these angles will be proportional to the lengths of the opposite sides; and the triangle may therefore be calculated as if it were a plane one.

All three methods were used in the French surveys. In the British Ordnance survey the triangles were generally

$$* \quad \text{chord} = \frac{\text{arc}^3}{24(\text{radius})^2}$$

calculated by the second method and checked by the third.

Legendre's theorem gives very nearly accurate results. In a triangle of which the sides were 220, 180, and 60 miles, the errors in the two long sides, as calculated by this method from the short side, would be only three-tenths of a foot.

The following investigation shows under what circumstances small errors in the measurements of the angles of a triangle have the least effect upon the calculated lengths of the sides.

Suppose that in a triangle $a\ b\ c$ we have the side b as a measured base, and measure the angles A and C; we have

$$a \sin B = b \sin A$$

If we suppose b to have been correctly measured we may treat it as a constant; and under this supposition if we differentiate the above equation we shall get

$$da = \frac{b \cos A}{\sin B} dA - \frac{a \cos B}{\sin B} d\,B$$

$$\text{or, since } \frac{b}{\sin B} = \frac{a}{\sin A}$$

$$da = a \cot A\, d\,A - a \cot B\, d\,B$$

d A and d B are here supposed to be positive, and represent small errors in the measurements of A and B. If they are assumed to be equal and of the same sign we shall have for the error of the side a,

$$da = a\, d\,A\ (\cot A - \cot B)$$

which becomes zero when A=B.

If d A and d B are supposed equal, but of opposite signs, we shall have

$$da = \pm\, a\, d\,A\ (\cot A + \cot B)$$

and since

$$\cot A + \cot B = \frac{\sin (A+B)}{\sin A \sin B} = \frac{\sin (A+B)}{\frac{1}{2}\cos (A-B) - \frac{1}{2}\cos (A+B)}$$

it follows that

$$da = \pm\, a\, d\, A \frac{2 \sin C}{\cos (A-B) + \cos C}$$

and *da* will be a minimum when A=B.

In either case we have the result that the best conditioned triangle is the equilateral.

CHAPTER IV.

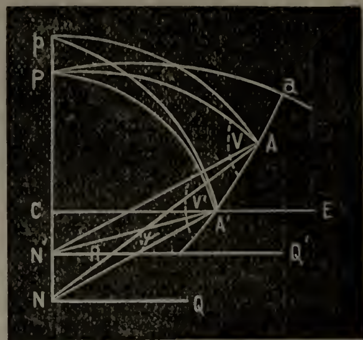
DETERMINATION OF THE GEODETIC LATITUDES, LONGITUDES, AND AZIMUTHS OF THE STATIONS OF A TRIANGULATION, TAKING INTO ACCOUNT THE ELLIPTICITY OF THE EARTH.

Where the lengths of all the sides of a triangulation have been computed it becomes necessary, in order to plot the positions of the stations on the chart, to obtain their latitudes and longitudes.

The first step to be taken is to determine by means of astronomical observations the true position of one of the stations, and also the azimuth of one of the sides leading from it. We can then, knowing the lengths of all the sides of the triangles and the angles they make with each other, deduce the azimuths of all the sides, and calculate the latitudes and longitudes of the other stations.

Before geodetical operations had been carried to the perfection they have now attained it was considered sufficient to solve this problem by the ordinary formulœ of spherical trigonometry, taking as the radius of the earth the radius at the mean latitude of the chain of triangles.

Thus in the triangle PAA′ (fig. 35) where P is the pole of the earth, and A, A′, two stations, if the latitude and longitude of A were known, and also the length and azimuth of A A′, we should have the two sides A P, A A′, and the included angle PAA′, and could use Napier's analogies to determine the remaining parts of the triangle, and thus obtain the latitude and longitude of A′, and the azimuth of A at A′. But this method is deficient in exactness, especially as regards the latitude, and the following has been adopted as giving better results.



(*Fig.* 35.)

Let A N be the normal at A, and suppose a sphere to be described with centre N and radius N A meeting the polar axis at p. Also let p A, p A′ be meridians on this sphere. We then calculate the geographical position of A′, not by the ordinary formulas of spherical trigonometry (since the side A A′ is very small relatively) but by the series

$$\text{I.}\quad a = b - c \cos A + \tfrac{1}{2} c^2 \cot b \sin_2 A + \tfrac{1}{2} c^3 \cos A \sin^2 A \left(\tfrac{1}{3} + \cot^2 b\right) + \ldots$$

$$\text{II.}\quad 180^\circ - B = A + c \sin A \cot b + \tfrac{1}{2} c^2 \sin A \cos A \,(1 + 2 \cot_2 b) + \tfrac{1}{3} c^3 \sin A \cos^2 A \cot b \,(3 + 4 \cot^2 b) - \tfrac{1}{6} c^3 \sin A \cot b \,(1 + 2 \cot^2 b) \ldots$$

$$\text{III.}\quad C = \frac{c}{\sin b} \sin A + \frac{c^2}{\sin b} \sin A \cos A \cot b + \tfrac{1}{3} \frac{c^3}{\sin b} \sin A \cos^2 A \,(1 + 4 \cot^2 b) - \tfrac{1}{3} \frac{c^3}{\sin b} \sin A \cot^2 b \ldots$$

Let L be the latitude of A
" L′ " " A′
" M be the longitude of A = A P*a*
" M′ " " A′ = A′ P*a*

Let AA′ = K, and let Z and Z' be the angles it makes with the meridians pA and pA′, respectively. Then, substituting this notation in the spherical triangle ABC, and expressing by u the value of K in terms of the radius, we have

$$a = 90^\circ - L' \qquad b = 90 - L$$
$$A = Z \qquad B = 180^\circ - Z'$$
$$c = u \qquad C = M' - M$$

which would be the values to introduce into the series I. II. III. ~~III, I, II~~; but in practice it is ~~more convenient~~ usual to count the azimuths from 0 to 360°, starting at the south and going round by the west, north, and east. ~~This makes~~ Let Z be the azimuth of A′ at A, and Z' the azimuth of A at A′. Then ~~Therefore~~ in Fig. 35 $V = 180^\circ - Z$, and $V' = 360^\circ - Z'$, and the series ~~I, II, III~~ will be changed respectively into

(*a*) $L' = L - u \cos Z - \frac{1}{2} u^2 \sin 1'' \sin^2 Z \tan L$

(*b*) $M' = M + \dfrac{u \sin Z}{\cos L} - \frac{1}{2} u^2 \sin 1'' \sin 2 Z \dfrac{\tan L}{\cos L}$

(*c*) $Z' = 180^\circ + Z - u \sin Z \tan L$
$+ \frac{1}{4} u^2 \sin 1'' \sin 2 Z (1 + 2 \tan^2 L)$

the arc u being supposed to be in seconds.

~~It sometimes happens that~~ the latitude L′ is not quite the true latitude of A′; for the latter is A′ N′ Q′, or the angle made by the normal A′ N′ with N′ Q′, while the latitude given by equation (*a*) is the angle A N Q. The correction of the latitude (ψ) is the angle N′ A′ N′; for A′NQ—A′N′Q′=A′RQ′—A′N′Q′=N′A′R

$$\text{and } \sin\psi = \frac{N\,N' \sin P\,N'\,A'}{N'\,A'}$$

To obtain ~~Before investigating~~ the exact value of this angle it should be noted that when the geodesic line K is more

than half a degree its amplitude in latitude on the sphere —say dL—becomes a different quantity—say $\triangle$ L—on the ellipsoid, and that these two amplitudes of arcs of the same length being inversely proportional to their radii of curvature N, R, we have

$$\triangle \mathrm{L} : d\mathrm{L} :: \mathrm{N} : \mathrm{R} :: 1 : \frac{1-e^2}{1-e^2 \sin^2 \mathrm{L}}$$

whence we have, very nearly

$$\triangle \mathrm{L} = d \mathrm{L} \, (1+e^2 \cos^2 \mathrm{L}), \text{ and consequently}$$

$$\psi = d \mathrm{L} \, e^2 \cos^2 \mathrm{L}$$

and therefore the corrected latitude L$'$ is

$$(a') \; \mathrm{L}' = \mathrm{L} - (u \cos Z + \tfrac{1}{2} u^2 \sin 1'' \sin^2 Z \tan \mathrm{L})(1 + e^2 \cos^2 \mathrm{L})$$

and we have in seconds,

$$u = \frac{\mathrm{K}\,(1-e^2 \sin^2 \mathrm{L})^{\frac{1}{2}}}{a \sin 1''} = \frac{\mathrm{K}}{\mathrm{N} \sin 1''}$$

The formulas (b) and (c) are not ordinarily used, for when the latitude L is known on the spheroid it is used to determine M$'$ and Z'. But in this case we must introduce L$'$ into the values of these two unknown quantities. Now we have the spherical triangle p A A$'$, giving

$$\sin (\mathrm{M}' - \mathrm{M}) = \frac{\sin u \sin \mathrm{Z}}{\cos \mathrm{L}'} \quad (\ell')$$

and, since u is very small,

$$\mathrm{M}' = \mathrm{M} + \frac{u \sin Z}{\cos \mathrm{L}'}$$

Also, in the same triangle

$$\cot \tfrac{1}{2} (\mathrm{A} + \mathrm{A}') = \tan \tfrac{1}{2} (\mathrm{M}' - \mathrm{M}) \frac{\sin \frac{1}{2} (\mathrm{L} + \mathrm{L}')}{\cos \frac{1}{2} (\mathrm{L} - \mathrm{L}')}$$

$$= \tan \left\{ 90^\circ - \frac{\mathrm{A} + \mathrm{A}'}{2} \right\}$$

but $90^\circ - \dfrac{\mathrm{A} + \mathrm{A}'}{2}$ and M$'$—M being always very small angles, and A + A$'$ being the same as Z'—Z, we have

$$(c') \; Z' = 180^\circ + Z - (\mathrm{M}' - \mathrm{M}) \frac{\sin \frac{1}{2} (\mathrm{L} + \mathrm{L}')}{\cos \frac{1}{2} (\mathrm{L} - \mathrm{L}')}$$

The imaginary sphere used in the above investigation will, of course, coincide with the spheroid for the parallel of latitude through the point A. Any plane passing through the normal will cut the surface of the sphere in the arc of a great circle, and the spheroid in a line, which, for about three degrees, will be practically a geodesic line.

The following is another way of treating the subject. Instead of taking the normal at one of the points A A′ as the radius of the imaginary sphere let us take the normal at the point B, mid-way between them, as in Fig. 36, and for the sake of simplicity let these points be on the same meridian. Let A N, A′ N′ be the normals at A A′, produce them to *Z* and *Z*′ respectively, and draw A *e*, A′ *e*′ parallel to the major axis O E. The astronomical latitudes of the two points are *Z* A *e*, *Z*′ A′ *e*′. If now we draw B C the

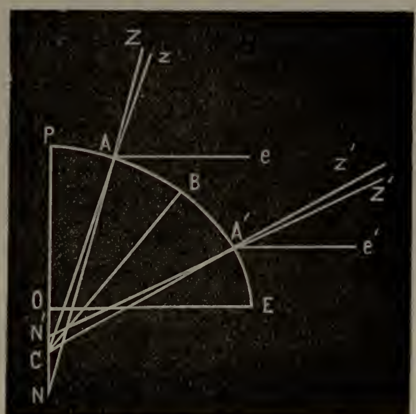


Fig. 36.

normal at B, C will fall between N and N′. The curve given in the figure is the elliptical meridian. The circular curve drawn with radius C B is not shown; but it would pass a little outside of A and A′. For practical purposes we may suppose it to pass through those points. Join C A, C A′, and produce them to *z* and *z*′ respectively. *z* A *e*, *z* A′ *e*′ will be the latitudes of A and A′ on the imaginary sphere, one being less and the other greater than the latitudes on the spheroid. The differences *Z* A *z*, *Z*′ A′ *z*′ may be considered the same. Let each be

designated $\frac{\delta}{2}$. Let L and L′ be the astronomical latitudes of A and A′, l, and l' their latitudes on the sphere, and λ the latitude of B. Then $\delta = L - L' - (l - l')$

$$\lambda = \frac{L+L'}{2} \text{ or } \frac{l+l'}{2}, \text{ and}$$

$$\frac{l-l'}{L-L'} = \frac{\text{radius of curvature at B}}{\text{normal at B}} = \frac{1-e^2}{1-e^2 \sin^2 \lambda}$$

$$\text{Also } \frac{l-l'}{L-L'} = \frac{L-L'-\delta}{L-L'} \qquad = \frac{1}{1+e^2 \cos^2 \lambda}$$

therefore

$$L-L'-\delta = \frac{L-L'}{1+e^2 \cos^2 \lambda}$$

$$\text{and } \delta = (L-L') \left\{ 1 - \frac{1}{1+e^2 \cos^2 \lambda} \right\}$$

$$= (L-L') \frac{e^2 \cos^2 \lambda}{1+e^2 \cos^2 \lambda}$$

$$= (L-L')\, e^2 \cos^2 \lambda, \text{ nearly.}$$

The angle δ is therefore nearly the same as the correction ψ already investigated.

In what next follows K is the distance A A′ in yards of any two stations A, A′, u the same distance in seconds of arc, R the radius of curvature of the meridian, N the normal (both in yards), e the eccentricity (=0.0817), and a the equatorial radius.

Equation (a') gives us the values of u and L′, (b') gives us M′, and (c') gives Z'. If we neglect the denominator of the fraction in (c') we have

$$Z' = 180° + Z - (M' - M) \sin \tfrac{1}{2} (L+L')$$

$$\text{or } Z' = 180° + Z - \frac{u'' \sin Z}{\cos L'} \sin \tfrac{1}{2} (L+L')$$

The last term of this equation, which is the difference of the azimuths at the two stations, is the convergence of their meridians.

If the triangulation is limited in extent it may be more convenient to express L′, M′, and Z' in terms of rectangular co-ordinates referred to axes having their origin at the station A, the axis of y being the meridian at A, and the axis of x the geodesic line through A and perpendicular to the meridian. The equations are

$$L' = L \pm \frac{y}{R \sin 1''} - \tfrac{1}{2} \sin 1'' \left\{ \frac{x}{N \sin 1''} \right\}^2 \tan \left(L \pm \frac{y}{R \sin 1''} \right)$$

$$M' = M \pm \frac{x}{N \sin 1''} \times \frac{1}{\cos L'}$$

$$Z' = 180^\circ + Z \pm \frac{x}{N \sin 1''} \tan L' \text{*}$$

The sphere described with radius equal to the normal for the mean latitude of two stations, A and B, may be used in the next three problems.

1. GIVEN THE LATITUDES AND LONGITUDES OF TWO POINTS TO FIND THE LENGTH AND DIRECTION OF THE LINE JOINING THEM.

Here we have given L, L′, M, and M′, and from L and L′ we obtain δ.

We have then to find l and l' from the equations

$$l = L - \frac{\delta}{2} \quad \text{and} \quad l' = L' + \frac{\delta}{2}$$

Let x'' be the number of seconds in the arc passing through the point of which L′ is the latitude and perpendicular to the meridian through the other point.

Let y'' be the number of seconds in the portion of this meridian between L and the foot of this perpendicular.

Let x, y, be the same quantities in linear units, N = the normal at the middle latitude, and Z = the azimuth of the point ~~L′~~ A′ from ~~L~~ A.

*These three formulæ, as well as the formulæ and examples given in the next few pages, have been taken from Jeffers' Nautical Surveying. Fractions of seconds have been omitted.

Then we shall have

$$x''=(M'-M)\cos l'$$
$$y''=l-l'-\tfrac{1}{2}\sin 1''\, x^{2''}\tan l$$
$$x=x''\, N \sin 1''$$
$$y=y''\, N \sin 1''$$
$$\tan Z=\frac{x''}{y''}$$
$$u''=\frac{x''}{\sin Z}=\frac{y''}{\cos Z}$$
$$K=u''\, N \sin 1''$$

The signs of (L—L′) and of (l—l') must be carefully attended to.

EXAMPLE.

Given

$$L=49^\circ\ 4'\ 25''$$
$$L'=49\ \ 22\ \ 33$$

M′—M, or difference of longitude$=38'\ 47''=2327''$
to find Z and K

$$\text{Here } L+L'=98^\circ\ 26'\ 58''$$
$$\lambda=\tfrac{1}{2}(L+L')=49\ \ 13\ \ 29$$
$$L'-L=\ 0\ \ 18\ \ 8$$
$$\tfrac{1}{2}(L'-L)=0\ \ 9\ \ 4=544''$$

To find the value of $\dfrac{\delta}{2}$

$$\log e^2=7.81085$$
$$\log \tfrac{1}{2}(L'-L)=2.73549$$
$$2\log\cos\tfrac{1}{2}(L+L')=9.62994$$

$$\log\frac{\delta}{2}=0.17628$$
$$\frac{\delta}{2}=-1''.5$$
$$l=L-\frac{\delta}{2}=49^\circ\ 4'\ 26''.5 \ (\delta \text{ being negative})$$
$$l'=L'+\frac{\delta}{2}=49^\circ\ 22'\ 31''.5$$

To find x''

$$\begin{aligned} \text{Log } (M'-M) &= 3.3668785 \\ \text{log cos } l' &= 9.8136470 \\ \hline \text{log } x'' &= 3.1805255 \\ x'' &= 1515'' \end{aligned}$$

To find the value of the 2nd term of y''

$$\begin{aligned} \text{log } \tfrac{1}{2} \sin 1'' &= 4.38454 \\ 2 \text{ log } x'' &= 6.36105 \\ \text{log tan } l &= 0.06197 \\ \hline \text{log 2nd term} &= 0.80756 \\ \hline \text{2nd term} &= 0^\circ\ 0'\ 6'' \\ l'-l &= 0\ \ 18\ \ 5 \\ \hline y'' &= 0\ \ 18\ \ 11 = 1091'' \end{aligned}$$

To find the azimuth Z

$$\begin{aligned} \text{Log } x'' &= 3.1805255 \\ \text{log } y'' &= 3.0378887 \\ \hline \text{log } \frac{x''}{y''} &= 0.1426368 \\ Z &= 125^\circ\ 45'\ 21'' \end{aligned}$$

To find log N sin 1″

$$\begin{aligned} \text{Log N (in yards)} &= 6.8443224 \\ \text{log sin } 1'' &= 4.6855749 \\ \hline \text{Log N sin } 1'' &= 1.5298973 \end{aligned}$$

To find log u''

$$\begin{aligned} \text{Log } y'' &= 3.0378887 \\ \text{log cos Z} &= 9.7666596 \\ \hline \text{log } u'' &= 3.2712291 \end{aligned}$$

To find K

$$\begin{aligned} \text{log } u'' &= 3.2712291 \\ \text{log N sin } 1'' &= 1.5298973 \\ \hline & 4.8011264 \\ K &= 63226 \text{ yards.} \end{aligned}$$

To find the co-ordinates.

Value of x.	Value of y.
$\text{Log } x''=3.1805255$	$\text{Log } y''=3.0378887$
$\log N \sin 1''=1.5298973$	$\log N \sin 1''=1.5298973$
$\log x=4.7104228$	$\log y=4.5677860$
$x=51336$ yards	$y=36965$ yards

TO COMPUTE THE DISTANCE BETWEEN TWO POINTS, KNOWING THEIR LATITUDES AND THE AZIMUTH OF ONE FROM THE OTHER.

Let L and L′ be the latitudes, Z the azimuth, and let

$$\frac{(L+L')}{2}=\lambda$$

Then we shall have, as before

$$\frac{\delta}{2}=\frac{e^2 (L-L') \cos^2 \lambda}{2}$$

$$N=\frac{a}{(1-e^2 \sin^2 \lambda)^{\frac{1}{2}}}$$

$$l=L-\frac{\delta}{2} \qquad l'=L'+\frac{\delta}{2}$$

Assume $$\frac{\tan l}{\cos Z}=\tan \varphi$$

$$\text{then, } \sin (\varphi-u'')=\frac{\sin l'}{\sin l} \sin \varphi$$

which gives u; and $K=u'' N \sin 1''$

The algebraic sign of cos Z will determine the sign of φ, and, consequently, whether u'' is to be added to or subtracted from φ.

Example—

$$\begin{aligned} L &= 49^\circ\ 4'\ 25''\ N \\ L' &= 49\ \ 22\ \ 33\ \ N \\ Z &= 125\ \ 45\ \ 21 \end{aligned}$$

Here, as in the last example, we find δ, and hence

$$\begin{aligned} l &= 49^\circ\ \ 4'\ 27'' \\ l' &= 49\ \ 22\ \ 32 \end{aligned}$$

To find the value of φ.

$$\log \tan l = 0.0619727$$
$$\log \cos Z = 9.7666566$$
$$\log \tan \varphi = 0.2953161$$
$$\varphi = -(63^\circ\ 7'\ 55'')$$

To find $\varphi - u$.

$$\log \sin \varphi = 9.9503895$$
$$\log \sin l' = 9.8802377$$
$$\text{co-log} \sin l = 0.1217320$$
$$\log \sin (\varphi - u) = 9.9523592$$
$$\varphi - u = -(63^\circ\ 39'\ 3'')$$
$$u = 0^\circ\ 31'\ 8'' = 1868'$$

From the equation $K = u'' N \sin 1''$ we find K to be 63228 yards.

TO COMPUTE THE DISTANCE BETWEEN TWO POINTS, KNOWING THE LATITUDE OF ONE, THE AZIMUTH FROM IT TO THE OTHER, AND THE DIFFERENCE OF THEIR LONGITUDES.

Using the same nomenclature as before, let L be the given latitude and m the difference of the longitudes:

$$\text{Take } L'' = L' + \delta$$
$$\text{Assume } \tan \varphi = \sin L \tan Z$$
$$\tan L'' = \tan L \cancel{\sin (\varphi - m)} \frac{\sin (\varphi - m)}{\sin \varphi}$$
$$\delta = e^2 (L - L'') \cos^2 \tfrac{1}{2}(L + L''), \text{ very nearly}$$
$$L' = L'' - \delta$$
$$l = L - \frac{\delta}{2}$$
$$l' = L' + \frac{\delta}{2}$$
$$u'' = \frac{m \cos l}{\sin Z} \qquad K = u'' N \sin 1''$$

The algebraic sign of tan Z will determine the sign of φ, and consequently whether it is to be increased or diminished by m.

Example—

$$\text{Let } L = 49^\circ\ 4'\ 25''\ \text{N}$$
$$Z = 125\ \ 45\ \ 21$$
$$m = 38'\ 47'' = 2327''$$

To find φ.

$$\begin{aligned}
\log\sin L &= 9.8782652\\
\log\tan Z &= 0.1426368\\
\hline
\log\tan\varphi &= 0.0209020\\
\varphi &= -(46^\circ\ 22'\ 42'')\\
m &= \quad 38\ \ 47\\
\hline
\varphi - m &= 47^\circ\ 1'\ 29''
\end{aligned}$$

To find L″

$$\begin{aligned}
\log\tan L &= 0.0619663\\
\log\sin(\varphi - m) &= 9.8643024\\
\text{co-log}\ \sin\varphi &= 0.1403154\\
\hline
\log\tan L'' &= 0.0665841\\
L'' &= 49^\circ\ 22'\ 30''
\end{aligned}$$

To find δ

$$\begin{aligned}
L &= 49^\circ\ \ 4'\ 25''\\
L'' &= 49\ \ 22\ \ 30\\
\hline
L - L'' &= -\ 18'\ 5'' = 1085''\\
L + L'' &= 98\ \ 26\ \ 55
\end{aligned}$$

$$\begin{aligned}
\log e^2 &= 7.81085\\
\log(L - L'') &= 3.03531\\
2\log\cos\tfrac{1}{2}(L + L'') &= 9.62994\\
\hline
\log\delta &= 0.47610\\
\delta &= \qquad -3''\\
L' = L'' - \delta &= 49^\circ\ 22'\ 33''\\
l = L - \frac{\delta}{2} &= 49^\circ\ \ 4'\ 26''.5\\
l' = L' + \frac{\delta}{2} &= 49^\circ\ 22'\ 31''.5
\end{aligned}$$

To find u'' and K—

$$\begin{aligned}
\log m &= 3.3668785\\
\log\cos l &= 9.8136471\\
\text{co-log}\ \sin Z &= 0.0907036\\
\hline
\log u'' &= 3.2712292\\
\log N \sin 1'' &= 1.5298973\\
\hline
\log K &= 4.8011265\\
K &= 63226 \text{ yards.}
\end{aligned}$$

On the North American boundary survey in 1845 the following method was employed to find the mutual azimuths of two distant points the latitudes and longitudes of which were known.

Let A and B be the two points, of which B is the northern, and P the pole. Then, treating the earth as a sphere, we have in the spherical triangle P A B the two sides P A, P B, and the angle A P B given, and have to find the angles A, B. This is done by the usual formulæ,

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{AP-BP}{2}}{\cos \frac{AP+BP}{2}} \times \cot \frac{P}{2}$$

$$\tan \tfrac{1}{2}(B-A) = \frac{\sin \frac{AP-BP}{2}}{\sin \frac{AP+BP}{2}} \times \cot \frac{P}{2}$$

which give $\frac{A+B}{2}$ and $\frac{B-A}{2}$

Then, $A = \frac{A+B}{2} - \frac{B-A}{2}$

$B = \frac{A+B}{2} + \frac{B-A}{2}$

To correct these azimuths for the earth's spheroidal form take $90°-A$ and $B-90°$, and calculate the angles α, β, from the formulæ

$$\sin \alpha = \frac{\sin AP}{\sqrt{75}} \qquad \sin \beta = \frac{\sin BP}{\sqrt{75}}$$

Then, if A' and B' are the true spheroidal azimuths,

$$\tan(90°-A') = \cos \alpha \tan(90°-A)$$

$$\tan(B'-90°) = \cos \beta \tan(B-90°)$$

This method is very useful when a long line has to be cut from one point to another through forests.

To find the accurate length of the arc on the surface of the earth between two very distant points of known latitude and longitude is a very difficult and not very useful problem. It is, however, often advisable to calculate the distances between stations that are within the limits of triangulation, as a check upon the geodesical operations; and in the case of an extended line of coast, or in a wild and difficult country where triangulation is impossible, this problem is most useful for the purpose of laying down upon paper a number of fixed points from which to carry on a survey.

In the triangle PAB mentioned in the last article we have, as before, the sides PA, PB, and the angle P, as data. By solving the triangle we obtain the length of the arc AB. If the azimuths can be observed at the two stations the accuracy of the result will be greatly increased, and we can obtain the difference of longitude of the two stations as follows :—It may be proved that the spherical excess in a spheroidal triangle is equal to that in a spherical triangle whose vertices have the same astronomical latitude and the same differences of longitude: from whence results the rule

$$\tan \frac{P}{2} = \frac{\cos \frac{PA - PB}{2}}{\cos \frac{PA + PB}{2}} \times \cot \frac{A+B}{2}$$

$$= \frac{\cos \frac{1}{2} \text{ diff. lat.}}{\sin \frac{1}{2} \text{ sum of lat.}} \times \cot \frac{A+B}{2}$$

which gives P, or the difference of longitude.

As a rule, a small error in the latitudes is of no importance unless the latitudes are small: but the azimuths must be observed with the greatest accuracy. The angle P being known we can get the length of the arc AB, and must then convert it into distance on the earth's surface, using the radius of curvature of the arc for the mean latitude.

It may be observed that when dealing with the sphere we have the definite equations of spherical trigonometry to work with; while, when a spheroidal surface is in question, we have only approximate formulæ. In most of the equations employed in higher geodesy the right hand side consists of the first few terms of a converging series, the remainder being so small that they may be omitted without causing any appreciable error. Thus, in calculating the length of a geodesic line between two distant points the smaller terms of the series might amount to only a few inches in 100 miles.

Captain Deville in his "Examples of Astronomic and Geodetic Calculations" gives some very simple methods of solving certain problems in Geodesy by means of tables of logarithms of the convergence of meridians for one ~~our~~ chain departure, and tables of the value of a chain in seconds of latitude and in seconds of longitude at different latitudes. By departure is meant, of course, the distance one point is east or west of the other. If a great circle (not being a meridian or the equator) is drawn on the earth's surface it will cut each meridian it crosses at a different angle according to the laitude of the point of section: in other words its azimuth is continually changing; and if we take two points, A and B, on this great circle, and P is the pole, the convergence between A and B will be, practically, 180°—(PAB+PBA.) If the two points are in the same latitude, and one chain distant from each other, each of the angles A and B will be less than 90° by half the convergence. If the distance be constant the convergence will increase as we recede from the equator (where it is nothing) towards the poles. In problems involving two stations of different latitude the convergence used it that for the mean of the two latitudes.

The examples given by Captain Deville are worked out by logarithms. In the following selection of problems the principle only of the method is indicated.

Prob. 1.—To find the convergence of meridians between two points of given latitude.

Here we have only to find by a traverse table the departure in chains and multiply it by the convergence for one chain for the mean latitude.

Prob. 2.—To refer to the meridian of a point B an azimuth reckoned from the meridian of another point A.

Calculate the convergence between the two points, and add or subtract it from the given azimuth according as B is east or west of A.

Prob. 3.—Given the latitude and longitude of a station A, and the azimuth and distance of another station B, to find the latitude and longitude of the latter.

The distance and azimuth being given we can find the departure and distance in latitude of B approximately by the traverse table, and have the approximate mean latitude. We next find the mean azimuth by multiplying the departure by the convergence for one chain at the mean latitude, and applying the convergence thus obained to the azimuth of B at A, which gives the azimuth of A at B, and hence the mean azimuth.

To get the correct latitude of B we multiply the distance by the cosine of the mean azimuth and by the value of one chain in seconds of latitude. This gives the difference of latitude of the two stations in seconds.

Similarly, the difference of longitude of the stations is found by multiplying the distance by the sine of the mean azimuth and by the value of one chain in seconds of longitude.

Prob. 4.—To correct a traverse by the sun's azimuth.

On a traverse survey of any extent the direction of the lines must be corrected from time to time by astronomical observations, usually either of the sun or the pole star.

If the traverse is commenced at a station A with a known orientation, and at another station B an observation is taken, the azimuth or bearing of the line thus obtained should differ from the azimuth as carried on through the traverse from A by the convergence of meridians between the stations. Should there prove to be any error it should be equally distributed among the courses by dividing it by the number of stations. Multiplying the result by the number of any course gives the correction for that course. As an example:—A traverse of seven courses was made in a westerly direction. At station 8, or the end of the 7th course, thc sun was observed, and its azimuth found to be 267 ~~270~~° 11′ 50″, the horizontal plate reading being 267° 59′ 10″. Required the error of orientation.

On calculating the convergence between stations 1 and 8 it was found to be 49′ 5″. If the traverse had been correctly run the sun's azimuth plus the convergence would have been the same as the plate reading; but the latter was 1′ 45″ too little. One seventh of this, or 15″, is the correction for each course, and we have to add 15″ to the plate reading of the first course, 30″ to that of the second, and so on.

267°	11′	50″
	49	5
268	0	55
267	59	10
	1	45

Prob. 5.—When running a line to correct its direction by the sun's azimuth.

Unless the line is a north and south one its azimuth will be continually changing from point to point. Its direction can be checked at any time by finding its azimuth astronomically to ascertain if this is what it ought to be after allowiug for the convergence. The first step is to find the approximate difference of latitude from the distance chained and the azimuth at which the line started. This will give the latitude of the station and

the mean latitude approximately. The latter being known, the azimuth and distance give the convergence, which being applied to the initial azimuth the true azimuth is obtained.

Prob. 6.—To lay out a given figure on the ground, correcting the courses by astronomical observations.

Take as an instance a square ABCD, the side AB being commenced at A with a given azimuth. The course is to be corrected by observations at the other three corners. The convergence between A and B being found in the usual manner and applied to the original azimuth (in addition to the angle at the corner) gives us the azimuth of BC. Similarly, the convergence between A and C will give us the azimuth of CD; and so on.

Prob. 7.—To lay out a parallel of latitude by chords of a given length.

The angle of deflection between two chords is the convergence of meridians for the length of a chord, and the azimuths will be 90° minus half the convergence and 270° plus half the convergence. The convergence is found in the usual way.

Prob. 8.—To lay out a parallel of latitude by offsets.

A parallel may be laid out by running a line perpendicular to a meridian and measuring offsets towards the nearest pole. The length of an offset is its distance from the meridian multiplied by the sine of half the convergence for that distance; or (since the distance is in this case the same as the departure) the square of the distance multiplied by the sine of half the convergence for one chain. As this angle is small the logarithm of its sine is obtained by adding the logarithm of the sine of half a second to the logarithm of the convergence for one chain departure.

When the offsets are equidistant any one of them may

be obtained by multiplying the first one by the square of the nnmber of the offset.

It is almost superfluous to point out that in practice all these problems are worked out by means of logarithms.

TO FIND THE AREA OF A PORTION OF THE SURFACE OF A SPHERE BOUNDED BY TWO PARALLELS OF LATITUDES AND TWO MERIDIANS (SPHERICAL SOLUTION.)

Let AB and CD be the meridians and AC, BD the parallels. Let φ be the latitude of A, φ' of B and $n°$ the difference of longitude of the meridians.

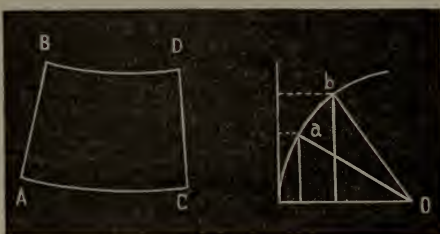


Fig. 37.

Now the area of the whole portion of the surface comprised between two parallels is equal to the area of the portion of the circumscribing cylinder (the axis of which is the polar axis) contained between the planes of the parallels produced to meet it. (*Vide* second figure showing a section, in which a is the point A and b the point B.)

Let r be the radius of the sphere and h the perpendicular distance between the planes.

Then the area of the spherical zone will be

$$
\begin{aligned}
&2\,\pi\, r \times h \\
&= 2\,\pi\, r \times r\ (\sin\varphi' - \sin\varphi) \\
&= 2\,\pi\, r^2\ (\sin\varphi' - \sin\varphi)
\end{aligned}
$$

$\therefore$ the area of the portion between the two meridians will be

$$\frac{n\,\pi\, r^2}{180}(\sin\varphi' - \sin\varphi)$$

TO FIND THE OFFSETS TO A PARALLEL OF LATITUDE.

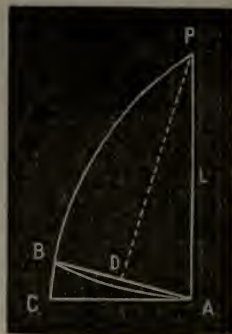


Fig. 38.

Let PA, PBC, be meridians, AB a portion of the parallel, AC a portion of a great circle touching the parallel at A.

It is required at a given latitude to find the offset BC for a given distance AC.

Let x be the circular measure of AC
" y do. do. BC
" l do. do. PA

AC and BC are very small.

In triangle PCA we have $\cos PC = \cos l \cos x$

$$= \cos l \left(1 - \frac{x^2}{2}\right) \text{ nearly.}$$

Therefore $\cos l - \cos PC = \cos l \frac{x^2}{2}$

$$\text{or } 2 \sin \frac{l + PC}{2} \sin \frac{BC}{2} = \cos l \frac{x^2}{2}$$

$$\text{or } 2 \sin l \frac{y}{2} = \frac{x^2}{2} \cos l, \text{ nearly.}$$

therefore, $y = \frac{1}{2} x^2 \cot l$.

(or, if x and y are measured lengths, and R is the radius of the earth, $y = \frac{x^2}{2R} \cot l$)

Next, join AB by a great circle arc. The angle BAC will be half the convergence, and AB=AC, approximately. Draw PD bisecting P, and therefore at right angles to AB.

In the triangle APD we have $D = 90°$

$$\text{and } PAD = 90° - \frac{\text{convergence}}{2}$$

Therefore, $\cos PAD = \tan AD \cot l$,

or $\sin \dfrac{\text{convergence}}{2} = AD \cot l$, approximately

$= \frac{1}{2} x \cot l$,,

Therefore, $y = \frac{1}{2} x^2 \cot l = x \sin \dfrac{\text{convergence}}{2}$

This is equally true if x and y are measured lengths.

CHAPTER V.

METHODS OF DELINEATING A SPHERICAL SURFACE ON A PLANE.

Since the surface of the globe is spherical, and as the surface of a sphere cannot be rolled out flat, like that of a cone, it is evident that maps of any large tract of country drawn on a flat sheet of paper cannot be made to exactly represent the relative position of the various points. It is necessary, therefore, to resort to some device in order that the points on the map may have as nearly as possible the same relative position to each other as the corresponding points on the earth's surface.

One method is to represent the points and lines of the sphere according to the rules of perspective, or as they would appear to the eye at some particular position with reference to the sphere and the plane of projection. Such a method is called a *projection*. The principal projections of the sphere are the "orthographic," "stereographic," "central or gnomonic" and "globular."

A second method is to lay down the points on the map according to some assumed mathematical law, the condition to be fulfilled being that the parts of the spherical surface to be represented, and their representations on the map, shall be similar in their small elements. To this

class belongs *Mercator's Projection*, in which the meridians are represented by equi-distant parallel straight lines, and the parallels of latitude by parallel straight lines at right angles to the meridians, but of which the distances from each other increase in going north or south from the equator in such a proportion as always to give the true bearings of places from one another.

The third method is to suppose a portion of the earth's surface to be a portion of the surface of a cone whose axis coincides with that of the earth, and whose vertex is somewhere beyond the pole, while its surface cuts or touches the sphere at certain points. The conical surface is then supposed to be developed as a plane, which it of course admits of being. The only conical development that will be discussed in these pages is the one known as the "ordinary polyconic."

The *Orthographic Projection* is simply the one employed in plans and elevations. When used for the delineation of a spherical surface the eye is supposed to be at an infinite distance, so that the rays of light are parallel, the plane of projection being perpendicular to their direction. In the case of a sphere the plane of projection is usually either that of the equator or of a meridian. When a hemisphere is projected on its base in this manner the relative positions of points near the centre are given with tolerable accuracy, but those near the circumference are completely distorted. The laws of this projection are easily deduced. Amongst others it is evident that in the case of a hemisphere projected on its base all circles passing through the pole of the hemisphere are projected as straight lines intersecting at the centre. Circles having their planes parallel to that of the base are projected as equal circles. All other circles are projected as ellipses of which the greater axis is equal to the diameter of the circle, and the lesser axis to the diameter multiplied by the cosine of the obliquity.

Stereographic Projection.—In this projection the eye is supposed to be situated at the surface of the sphere, and the plane of the projection is that of the great circle which is every where 90 degrees from the position of the eye. It derives its name from the fact that it results from the intersection of two solids, the cone and the sphere. Its principal properties are the following: 1. The projection of any circle on the sphere which does not pass through the eye is a circle; and circles whose planes pass through the eye are straight lines. 2. The angle made on the surface of the sphere by two circles which cut each other and the angle made by their projections are equal. 3. If C is the pole of the point of sight and c its projection; then any point A is projected into a point a such that $c\,a$ is equal to

$$r \times \frac{\cancel{\tan \text{arc } CA}}{2} \qquad \tan \frac{\text{arc } CA}{2}$$

where r is the radius of the sphere. From the second property it follows that any very small portion of the spherical surface and its projection are similar figures; a property of great importance in the construction of maps, and one which is also shared by Mercator's projection.

The astronomical triangle PZS can evidently be easily drawn on the stereographic projection. Z will be the pole of the point of sight. The lengths of ZP and ZS are straight lines found by the rule given above, and the angle Z being known the points P and S are known. The angles P and S being also known we can draw the circular curve PS by a simple construction.

The orthographic and stereographic projections were both employed by the ancient Greek astronomers for the purpose of representing the celestial sphere, with its circles, on a plane.

Gnomonic or Central Projection.—In this case the eye is at the centre of the sphere, and the plane of projection is

a plane touching the sphere at any assumed point. The projection of any point is the extremity of the tangent of the arc intercepted between that point and the point of contact. As the tangent increases very rapidly when the arc is more than 45°, and becomes infinite at 90°, it is evident that this projection cannot be adopted for a whole hemisphere.

Globular Projection.—This is a device to avoid the distortion which occurs in the above projections as we approach the circumference of the hemisphere. In the accompanying figure let A C B be the hemisphere to be represented on the plane A B, E the position of the eye, O the centre of the sphere, and EDOC perpendicular to the plane A B. M and F are points on the sphere, and their projections are N and G. Now the representation would be perfect if A N : N G : G O were as A M : M F : F C. This cannot be obtained exactly, but it will be approximately so if the point E is so ~~shown~~ taken that G is the middle point of A O and F the middle point of A C. In this case, by joining F O and drawing F L perpendicular to O C, it may easily be shown that E D is equal to O L, which is O F × cos 45°, or $r \times 0.71$ nearly—~~because~~ for since G O is half the radius and F L half the inscribed square—~~therefore~~

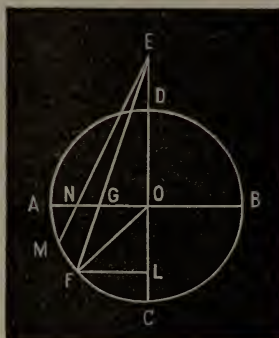


Fig. 39.

F L : G O :: O C : O L

but F L : G O :: L E : O E

∴ L E : O E :: O C : O L

consequently, LO : O E :: C L : O L, or $OL^2 = OE.CL$

but $OL^2 = FL^2 = DL.LC$, ∴ $OE.CL = DL.CL$

or $OE = DL$

that is, $ED = OL$

The above projections are seldom used for delineating the features of a single country or a small portion of the earth's surface. For this purpose it is more convenient to employ one of the methods of development.

Mercator's Projection is the method employed in the construction of nautical charts. The meridians are represented by equi-distant parallel straight lines, and the parallels of latitude by straight lines perpendicular to the meridians. As we recede from the equator towards the poles the distances between the parallels of latitude on the map are made to increase at the same rate that the scale of the distance between points east and west of each other increases on the map, owing to the meridians being drawn parallel instead of converging. If we take l as the length of a degree of longitude at the equator (which would be the same as a degree of latitude supposing the earth a sphere), and l' that of a degree of longitude at latitude λ, then $l'=l \cos \lambda$, or $l' \cdot\cdot l :: 1 : \sec \lambda$. Now $l' : l$ is the proportion in which the length of a given distance in longitude has been increased on the map by making the meridians parallel, and is therefore the proportion in which the distance between the parallels of latitude must be increased. It is evident that the poles can never be shown on this projection, as they would be at an infinite distance from the equator.

If a ship steers a fixed course by the compass this course is always a straight line on a Mercator's chart. Great circles on the globe are projected as curves, except in the case of meridians and the equator.

In this projection, though the scale increases as we approach the poles, the map of a limited tract of country gives places in their correct relative positions.

The Ordinary Polyconic Projection.—In conical developments of the sphere a polygon is supposed to be inscribed in a meridian. By revolution about the polar axis the

polygon will describe a series of frustums of cones. If the arc of the curve equals its chord the two surfaces will be equal. In this manner the spherical surface may be looked upon as formed by the intersection of an infinite number of cones tangential to the surface along successive parallels of latitude. These conical surfaces may be developed on a plane, and the properties of the resulting chart will depend on the law of the development.

The Ordinary Polyconic is a projection much used in the United States Coast Survey. It is peculiarly applicable to the case where the chart embraces considerable difference in latitude with only a moderate amplitude of longitude, as it is independent of change of latitude.

Before describing it it must be noted that whatever projection is used the spheroidal figure of the earth must be taken into account, its surface being that which would be formed by the revolution of a nearly circular ellipse round the polar axis as a minor axis.

In the Ordinary Polyconic each parallel of latitude is represented on a plane by the development of a cone having the parallel for its base, and its vertex at the point where a tangent to a meridian at the parallel intersects the earth's axis, the degrees on the parallel preserving their true length. A straight line running north and south represents the middle meridian on the chart, and is made equal to its rectified arc according to scale. The conical elements are developed equally on each side of this meridian, and are disposed in arcs of circles described (in the case of the sphere) with radii equal to the radius of the sphere multiplied by the cotangent of the latitude. The centres of these arcs lie in the middle meridian produced, each arc cutting it at its proper latitude.

These elements evidently touch each other only at the middle meridian, diverging as they leave it. The curva-

ture of the parallels decreases as the distance from the poles increases, till at the equator the parallel becomes a straight line.

To trace the meridians we set off on the different parallels (according to the usual law for the length of an arc of longitude) the true points where each meridian cuts them, and draw curves connecting those points.

To allow for the ellipticity of the earth we must use for the radius of the developed parallel N cot l, where

$$N = \frac{a}{(1 - e^2 \sin^2 l)^{1/2}}$$

a being the equatorial radius, e the eccentricity, N the normal terminating in the minor axis, and l the angle it makes with the major axis.

It is evident that the slant height of the cone—say r—is N cot l, and that the radius of the parallel on the spheroid is N cos l. The length of an arc of $n°$ of a parallel will be $n° \frac{\pi}{180°}$ N cos l.

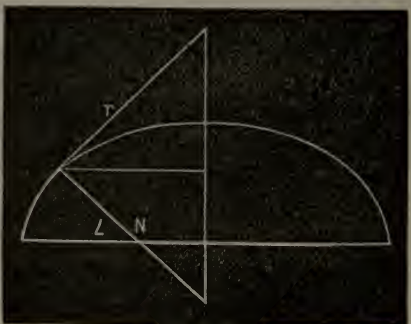


Fig. 40.

In practice, instead of describing the arcs of the parallels with radii, it is more convenient to construct them from their equations as circles. The intersections of the meridians and parallels can also be found in this way. Express x and y, the rectangular co-ordinates of a point, as functions of the radius of the developed parallel (N cot l) and the angle (θ) that this radius makes with the middle meridian.

Take the origin at L (Fig. 41) the point of intersection of a parallel with the middle meridian; the middle meridian as the axis of y; and the perpendicular through L as the axis of x. Then we shall have for any point P whose latitude is l and longitude from the meridian $n°$

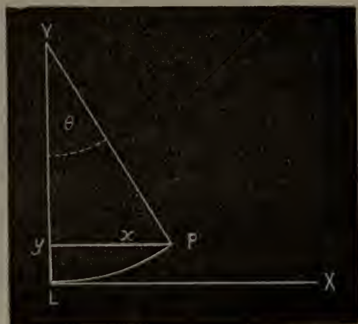


Fig. 41.

$$x = \mathrm{Y\,P} \sin\theta = \mathrm{N} \cot l \sin\theta \qquad (1)$$

$$y = \mathrm{Y\,P} \operatorname{versin}\theta = \mathrm{N} \cot l \operatorname{versin}\theta \qquad (2)$$

θ being, of course, some function of n.

To find the relation between θ and n, since the parallels are developed with their true lengths the distance LP equals the length of the portion LP of the parallel on the spheroid. Therefore the angles at the centres of the two arcs will be inversely proportional to the radii, and

$$\frac{\mathrm{N} \cot l}{\mathrm{N} \cos l} = \frac{n°}{\theta}, \text{ or } \theta = n° \sin l \qquad (3)$$

These three equations are sufficient to project any point of the spheroid when we know its latitude and its longitude from the middle meridian. If we take n constant we can project the successive points of any meridian.

If S is the distance on the elliptical middle meridian from the origin to the point where the parallel through the point to be projected cuts the middle meridian, equation (2) will become, $y = \mathrm{N} \cot l \operatorname{versin}\theta \pm \mathrm{S}$.

From the above equations tables may be formed for the construction of charts.

Fig. 42 shows the geometrical relation between the angles θ and n.

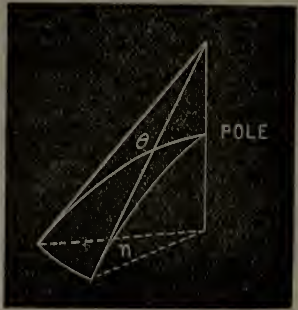


Fig. 42

This projection, when the amplitude in longitude does not exceed three degrees from the middle meridian, has the following properties.

It distorts very little, and has great uniformity of scale.

It is well adapted to all parts of the earth, but best to the polar regions.

The meridians make practically the same angles with each other and with the parallels as on the sphere. Angles are projected with little change.

The great circle or geodesic line is projected as a straight line practically equal to itself.

CHAPTER VI.

TRIGONOMETRICAL LEVELLING.

TO FIND THE HEIGHT OF A POINT B ABOVE A STATION A.

In the accompanying figure O is the centre of the earth, AC is tangential to the earth's surface at A, B′ is the apparent position of B, owing to refraction. CC′ is the correction for curvature, or $\frac{K^2}{2R}$, where K is the horizontal distance of B from A, and R is the radius of the earth; both in feet. BB′ is about 0.16 CC′

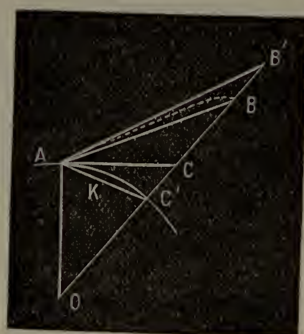


Fig. 43.

If the distances K is not very great

ACB may be taken as a right angle, and AC, the arc AC′, and the straight line AC′, are all practically equal. We shall have then, if ~~the distance K is not very great,~~

$$BC' = K \tan B'AC + CC' - BB'$$
$$= K \tan B'AC + 0.00000002\, K^2$$

where B′AC is the observed angle of elevation of B. This formula supposes that AC′B is practically 90°. If the dis-

tance is so great that this is not the case we shall have in the triangle ACB

$$BC = K\frac{\sin BAC}{\sin B}$$

To find the angle B, we have in the triangle AOB,

$$\begin{aligned} B &= 180° - (O + BAO) \\ &= 180° - (O + 90° + BAC) \\ &= 90° - (O + BAC) \end{aligned}$$

Hence, $\sin B = \cos (O + BAC)$

$$BC = K \times \frac{\sin BAC}{\cos (O + BAC)}$$

$$\text{And } BC' = BC + CC' = BC + \frac{K^2}{2R}$$

(where R is the radius of the earth in feet.)

The angle O, in minutes, is 0.0001646 K, and

$\frac{K^2}{2R}$ is $0.000000023936\ K^2$

RECIPROCAL OBSERVATIONS FOR CANCELLING REFRACTION.

If we measure the reciprocal angles of elevation and depression of two stations—in other words, if at each we observe the zenith distance of the other—we shall get rid of the effects of refraction. Let α be the angle of elevation of B at A and β the angle of depression of A at B.

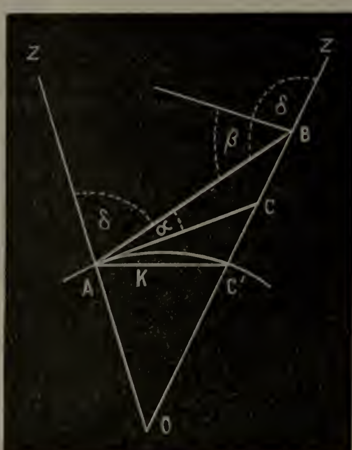


Fig. 44.

Then

$$BC' = K \times \frac{\sin \frac{1}{2} (\alpha + \beta)}{\cos \frac{1}{2} (\alpha + \beta + O)}$$

If the zenith distances are observed call them δ and δ', and we shall have (since $\delta = 90° - \alpha$ and $\delta' = 90° + \beta$)

$$BC' = K \times \frac{\sin \frac{1}{2}(\delta' - \delta)}{\cos \frac{1}{2}(\delta' - \delta + O)}$$

If O is very small compared with the other angles we may neglect it, when we shall have

$$\begin{aligned} BC' &= K \tan \tfrac{1}{2}(\alpha + \beta) \\ &= K \tan \tfrac{1}{2}(\delta' - \delta)^{*} \end{aligned}$$

REDUCTION TO THE SUMMITS OF THE SIGNALS.

Suppose there are two stations, *a* and *b*, which cannot be seen from each other, so that signals have to be erected at each. Let A and B be the summits of the signals, α and β the true angles of elevation [and depression of *a* and *b* respectively. At *a* the angle B *a* C is observed and at *b* the angle A *b* D. Call B *a* C, θ; A *b* D, ψ; A *a*, *h*; and B *b*, *h'*. Then, to find the reduced angles α and β we shall have

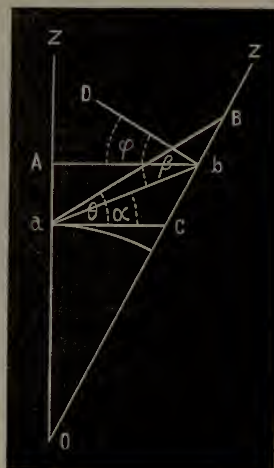


Fig. 45.

$$\alpha = \theta - \frac{h \cos \cancel{\psi}\, \theta}{K \sin 1''}$$

$$\beta = \varphi + \frac{h' \cos \psi}{K \sin 1''}$$

the differences being in seconds.

If zenith distances $\triangle$ and $\triangle'$ are taken we shall have

---

*Clarke gives the formula

$$h' - h = K \tan \tfrac{1}{2}(\delta' - \delta) \left\{ 1 + \frac{h + h'}{2r} \right\}$$

where *h* and *h'* are the heights of the stations, and *r* the radius of the earth.

for δ and δ'

$$\delta = \Delta + \frac{h \sin \Delta}{K \sin 1''}$$

$$\delta' = \Delta' + \frac{h' \sin \Delta'}{K \sin 1''}$$

Reciprocal observations ought to be simultaneous in order that the effects of refraction may be as nearly as possible the same for both.

In problems of this kind we ought, strictly speaking, instead of using the mean radius of the earth, to take the normal for the mean latitude of the stations.

The following geodetical formulœ are used for more exact determinations. In addition to the letters used in the foregoing problems we have a the known altitude of the lower station; N the normal for the mean latitude; M the modulus of common logarithms; and r the coefficient of refraction.

1. TO FIND THE DIFFERENCE OF LEVEL BY RECIPROCAL ZENITH DISTANCES.

$$\text{Log. diff. of level} = \log\left\{K \tan \tfrac{1}{2}(\delta' - \delta)\right\} + \frac{M}{N} a \pm \frac{M}{2N} K \tan \tfrac{1}{2}(\delta' - \delta) + \frac{M}{12 N^2} K^2$$

2. TO FIND THE DIFFERENCE OF LEVEL BY MEANS OF A SINGLE ZENITH DISTANCE.

$$\text{Log. diff. level} = \log.\left\{\frac{K}{\tan\left\{\delta - \dfrac{1 - 2r}{2 N \sin 1''} K\right\}}\right\}$$

$$+ \frac{M}{N} a \pm \frac{M}{2N} \frac{K}{\tan\left\{\delta - \dfrac{1 - 2r}{2 N \sin 1''} K\right\}} + \frac{M}{12 N^2} K^2$$

The third term is positive if Δ is less than 90°.

3. TO ASCERTAIN THE HEIGHT OF A STATION BY MEANS OF THE ZENITH DISTANCE OF THE SEA HORIZON.

In this case, when possible, different points of the horizon should be observed on different days and the mean of the whole taken, the state of the tide being also noted.

The formula is

$$\text{Log. altitude} = \log \frac{N}{2}\left\{\frac{\sin 1''}{1-r}\right\}^2 + \log.\ (\delta - 90^\circ)^2$$

$$+\frac{M}{4}\left\{\frac{\sin 1''}{1-r}\right\}^2 (\delta - 90^\circ)^2$$

The angle δ—90° is in seconds $(\delta - 90^\circ)^2$

The last term may generally be neglected.

The following is an example of finding the difference of level by a single zenith distance.

The altitude of the lower station (*a*) was 1000 yards, and *h* or the height of the instrument 5.

The horizontal distance between the stations (K) was 57836 yards. The zenith distance of the upper station (Δ) 88° 24′ 40″.

First, to find the value of the angle δ.

$$\begin{aligned} \text{Log } h &= 0.69897 \\ \text{Log sin } \Delta &= 9.99984 \\ \text{Co-log K} &= 5.23780 \\ \text{Co-log sin } 1'' &= 5.31443 \end{aligned}$$

$$\text{Log } \frac{h \sin \Delta}{K \sin 1''} = 1.25104 = \log 17''.8$$

Therefore $\delta = 88^\circ\ 24'\ 57''.8$

Next, to find the value of the angle $\frac{1-2r}{2\,N \sin 1''}K$

$$\text{Log } \frac{1-2r}{2\,N \sin 1''} = 8.13252$$

$$\text{Log K} = 4.76220$$

$$2.89472 = \log 784''.7 = 0^\circ\ 13'\ 4''.7$$

$$\delta - \frac{1-2r}{2\,N \sin 1''}K = 88^\circ\ 11'\ 53''.1$$

Thirdly, value of the difference of level.

Log K	=4.7621984
Log tan 88″ 11′ 53″.1	=1.5022427
Log 1st term	=3.2599557
2nd term	= 691
3rd term	= +627
4th term	= 9

Log. diff. level=3·2600884=log 1820.07 yards.

Second Term.

$\text{Log}\,\dfrac{M}{N} = 2{\cdot}8393$

Log a=3

Log 2nd term=5·8393

2nd term=0.0000691

Third Term.

$\text{Log}\,\dfrac{M}{2\,N} = 2{\cdot}5383$

log 1st term=3·2599

log 3rd term=5·7982

3rd term=0·0000627

Fourth Term.

$\text{Log}\,\dfrac{M}{12\,N^2} = 3{\cdot}4387$

log K2 =9·5244

log 4th term=2·9631

4th term=0·0000009

REFRACTION, &c.

TO FIND THE CO-EFFICIENT OF TERRESTRIAL REFRACTION BY RECIPROCAL OBSERVATIONS OF ZENITH DISTANCES.

Let A and B be two stations, and let their heights (ascertained by levelling) be h and h'. Consider the earth as a sphere, and take O′ its centre. Call the radius r and the angle AOB v. Let Z be the true zenith distance of B at A, viz., ZAB, and Z′ that of A at B or Z′BA. The dotted curve shows the path of the ray of light. A′ and B′ are the apparent positions of the stations.

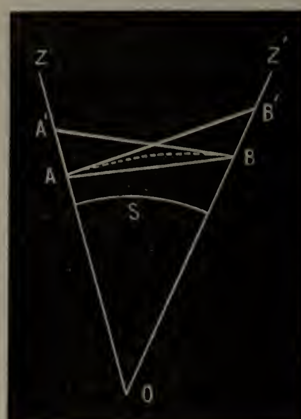


Fig. 46.

The co-efficient of refraction is the ratio of the difference between the observed and real zenith distance at

either station to the angle v. Thus, if k is the co-efficient and $z\ z'$ the observed zenith distances, we have k equal to $\frac{Z-z}{v}$ or $\frac{Z'-z'}{v}$. But these are not always the same.

In the triangle ACB we have

$$\tfrac{1}{2}(Z'+Z)=90^\circ+\frac{v}{2}$$

$$\tan\frac{v}{2}\tan\tfrac{1}{2}(Z'-Z)=\frac{h'-h}{h'+2r+h}$$

These equations give Z′ and Z.

If we substitute for $\tan\frac{v}{2}$ the first two terms of its expansion in ~~sines~~ series, the second equation may be put in the form

$$h'-h=s\tan\tfrac{1}{2}(Z'-Z)\left\{1+\frac{h+h'}{2r}+\frac{s^2}{12r^2}\right\}$$

where s is the length of AB projected on the sea level.

The co-efficient of refraction may also be obtained from the simultaneous reciprocally-observed zenith distances of A and B without knowing their heights. Thus:

$$Z=z+k\,v,\text{ and }Z'=z'+k\,v$$
$$\therefore\ z+z'+2\,k\,v=180^\circ+v$$
$$\text{or }1-2\,k=\frac{z+z'-180^\circ}{v}$$

The mean co-efficient is .0771. For rays crossing the sea it is .0809, and for rays not crossing it .0750.

The amount of terrestrial refraction is very variable, and not to be expressed by any single law. In flat, hot countries where the rays of light have to pass near the ground and through masses of atmosphere of different densities the irregularity of the refraction is very great: so much so that the path of the rays is sometimes convex to the surface of the earth instead of concave. In Great Britain the refraction is, as a rule, greatest in the early mornings; towards the middle of the day it decreases and remains nearly constant for some hours, increasing again towards evening.

CHAPTER VII.

THE USE OF THE PENDULUM IN DETERMINING THE COMPRESSION OF THE EARTH.

The spheroidal form of the earth causes the force of gravity to increase from the equator towards the poles, and this force may be measured at any place by means of the oscillations of a pendulum.

If we had a heavy particle suspended from a fixed point by a fine inextensible thread without weight we should have what is called a *simple* pendulum. If this pendulum were allowed to make small oscillations (of not more than a degree in amplitude) *in vacuo*, and in a vertical plane, the time of oscillation would be given by the formula

$$t = \pi\left\{\frac{l}{g}\right\}^{\frac{1}{2}} \qquad (1)$$

Where t is the number of seconds, l the length of the pendulum in feet, and g the force of gravity.

Therefore, taking g as constant, if there were another pendulum l' feet long and vibrating in t' seconds, we should have

$$t : t' :: \sqrt{l} : \sqrt{l'}$$

or, if the time were constant and g changed to g',

$$\cancel{t : t' :: g' : g}$$

$$l : l' :: g : g'$$

or if l were constant and g variable

$$t : t' :: \frac{1}{\sqrt{g}} : \frac{1}{\sqrt{g'}} \qquad (2)$$

If n and n' are the number of oscillations in ~~the~~ a time for a
t and t', then, $n' : n :: t : t'$

$$:: \sqrt{l} : \sqrt{l'}$$

From (2) we have, $g' = \frac{t^2}{t'^2} g$

$$= \frac{n'^2}{n^2} g \qquad (3)$$

To find the value of g' we can either ascertain by measurement the length of a pendulum that makes a certain number of oscillations in a given time, or we can use a pendulum of invariable length and find g' from equation (3). Both methods have been used, but the last is the easiest in practice.

A simple pendulum as described above is, of course, an imaginary quantity, and all pendulums actually used are what are called "compound" pendulums. But it is possible to calculate the length l of a simple pendulum that would oscillate in the same time as the compound one, by finding the position of the "centre of oscillation;" that is, of the point which moves in the same manner as would the pendulum if its whole mass were collected at that point, thus constituting a simple pendulum. The centres of oscillation and suspension are interchangeable, and if a pendulum is suspended from the former, the latter becomes the new centre of oscillation.

The compression of the earth is calculated thus: If c is the compression, φ the latitude of a station, g the force of gravity at the equator, g' that at the station, and m the ratio of the centrifugal force at the equator to g, we have, by the formula known as Clairaut's Theorem,

$$g' = g \; [1 + (\tfrac{5}{2}\, m - c) \sin^2 \varphi]$$

and, since $g' = \frac{n'^2}{n^2}\, g$, if n is the number of oscillations in

a given time at the equator and n' the number at the station,

$$n'^2 = n^2 \left[1 + \left(\tfrac{5}{2}m - c\right) \sin^2 \varphi\right) \qquad (4)$$

Also, $g' = \frac{l'}{l} g$

∴ if we take the lengths of the seconds pendulums instead of the number of their oscillations, we have

$$l' = l \left[1 + \left(\tfrac{5}{2}m - c\right) \sin^2 \varphi\right] \qquad (5)$$

l being the length of the pendulum at the equator. m being known, and $n\ n'$, or $l\ l'$, being found by experiment, we at once get the value of c from equation (4) or (5).

Borda's pendulum, which was used by the French astronomers to find the length of the second's pendulum (that is, a pendulum oscillating in a single second) at different stations, consisted of a sphere of platinum suspended by a fine wire, attached to the upper end of which was a knife edge of steel resting on a level agate plane. The length of the simple pendulum corresponding to Borda's was obtained by measurement and calculation.

In 1818 Captain Kater determined the length of the seconds pendulum in London (39.13929 inches) by means of a pendulum which had two knife edges facing each other—one for the centre of suspension, the other at the centre of oscillation—so that, provided the two knife edges were at the correct distance apart, they could be used indifferently as points of suspension; the pendulum being, of course, inverted in the two positions. The pendulum was made to swing equally from either point of suspension by adjusting a sliding weight. The distance between the two edges gave the length of the simple pendulum.

The advantage of such a pendulum is that it contains two in one, and that any injury to the instrument is detected by its giving different results when swung in the two positions. This pendulum was afterwards superseded by another of similar principle, in which, instead of

using a sliding weight, one end of the bar of which it consisted was filed away until the vibrations in the two positions were synchronous. In using the pendulum it is swung in front of the pendulum of an astronomical clock, the exact rate of which is known. By means of certain contrivances the number of vibrations made by the two pendulums in a given time can be compared exactly, and the number made by the clock being known that of the experimental pendulum is obtained. Certain corrections have to be applied. One for changes in the thermometer, which lengthen or shorten the pendulum : a second for changes in barometric pressure, which by altering the floatation effect of the atmosphere on the instrument, affect the action of gravity on it ; a third for height of station above the sea level, which also affects the force of gravity, the latter diminishing with the square of the distance from the centre of the earth ; and a fourth for the amplitude of the arc through which the pendulum swings, which, in theory, should be indefinitely small.

The number of pendulum oscillations in a given time has been observed at a vast number of stations in various parts of the world, and in latitudes from the equator to nearly 80°. The most extensive series of observations was one lately brought to a close in India, the pendulums used in which had been previously tested at Kew. The general results of all the pendulum experiments gives about 292 : 293 as the ratio of the earth's axes, which is the same as that deduced from measurements of meridianal arcs.